Signals & Transforms — TEST 2
(part of AM module 4 — 201800138)

Date:	04-06-2019
Place:	Therm
Time:	8:45–10:15 (till 10:45 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

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1. The *Hilbert transform* of a real signal $f(t)$ is the signal $g(t)$ whose Fourier transform equals $-\mathrm{i}\hat{f}(\omega)\,\mathrm{sgn}(\omega)$. Here $\mathrm{sgn}(\omega) = 1$ if $\omega > 0$, and $\mathrm{sgn}(\omega) = -1$ if $\omega < 0$, and $\mathrm{sgn}(0) = 0$.

   (a) Show that $g(t)$ is a real signal if $f(t)$ is a real signal.

   (b) Determine the Hilbert transform of $f(t) = \cos(t)$.

   (c) Suppose $f(t)$ has finite energy. Express the energy of $g(t)$ in terms of the energy of $f(t)$.

2. Determine the convolution of $f(t) = \mathrm{sinc}(t-1)$ and $g(t) = \mathrm{sinc}(2t)$ via Fourier or Laplace transformation.

3. Given is the differential equation

$$y^{(2)}(t) + 4y^{(1)}(t) + 4y(t) = u^{(1)}(t) - u(t). \tag{1}$$

   (a) Determine the impulse response of (1).

   (b) Let $u(t) = \mathbb{1}(t)$ and assume that $y(t) = 0$ for all $t \leq 0$. Determine the solution $y(t)$ of (1) for all $t \in \mathbb{R}$.

   (c) Suppose that $u(t) = \mathrm{e}^t\,\mathbb{1}(t)$. Determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 0$ and $y^{(1)}(0^-) = -3$.

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problem:	1	2	3
points:	4+4+3	6	3+4+3
	11	6	10

Test grade is $1 + 9p/p_{\max}$

Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\hat{f}\left(\frac{\omega}{a}\right)$	$a \in \mathbb{R}, a \neq 0$
Time-shift	$f(t-\tau)$	$\hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$	
Frequency-shift	$f(t)\mathrm{e}^{\mathrm{i}\omega_0 t}$	$\hat{f}(\omega-\omega_0)$	
Modulation Thm.	$f(t)\cos(\omega_0 t)$	$\frac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$	
Differentiation (time)	$f^{(1)}(t)$	$(\mathrm{i}\omega)\hat{f}(\omega)$	$\lim_{t\to\pm\infty} f(t) = 0$
Integration (time)	$\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\hat{f}(0) = 0$
Differentiation (freq.)	$-\mathrm{i}tf(t)$	$\hat{f}'(\omega)$	

$f(t)$	$\hat{f}(\omega)$	Condition
$\mathrm{rect}_a(t)$	$a\,\mathrm{sinc}(a\omega/2)$	$a > 0$
$\mathrm{trian}_a(t)$	$a\,\mathrm{sinc}^2(a\omega/2)$	$a \in \mathbb{R},\ a > 0$
$\mathrm{e}^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$\mathrm{Re}(a) > 0$
$\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t)$	$\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}(a) > 0;\ n \in \mathbb{N}$
$-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t)$	$\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}(a) < 0;\ n \in \mathbb{N}$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}\mathrm{e}^{-\omega^2/(4a)}$	$a \in \mathbb{R},\ a > 0$
$a\,\mathrm{sinc}(at/2)$	$2\pi\,\mathrm{rect}_a(\omega)$	$a \in \mathbb{R},\ a > 0$

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t-b)$	$\mathrm{e}^{-\mathrm{i}\omega b}$
$\mathrm{e}^{\mathrm{i}\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
$\mathrm{sgn}(t)$	$\frac{2}{\mathrm{i}\omega}$
$\mathbb{1}(t)$	$\frac{1}{\mathrm{i}\omega} + \pi\delta(\omega)$

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time-scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$ (if $a > 0$)
Time-shift	$f(t-t_0)\,\mathbb{1}(t-t_0^-)$	$F(s)\mathrm{e}^{-st_0}$ (if $t_0 > 0$)
Shift in s-domain	$f(t)\mathrm{e}^{s_0 t}$	$F(s-s_0)$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$
	$f^{(2)}(t)$	$s^2F(s) - sf(0^-) - f^{(1)}(0^-)$
Integration (t)	$\int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau$	$\frac{F(s)}{s}$
Differentiation (s)	$-tf(t)$	$F'(s)$

$f(t),\ (t > 0^-)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
$\frac{t^n}{n!}$ $(n \in \mathbb{N})$	$\frac{1}{s^{n+1}}$
$\frac{t^n}{n!}\mathrm{e}^{at}$ $(n \in \mathbb{N})$	$\frac{1}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\mathrm{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathrm{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\delta(t)$	1