Signals & Transforms (202001343) — TEST 1

Date: 17-05-2021
Place: NH 115
Time: 09:00–10:30 (till 10:55 for students with special rights)
Course coordinator: G. Meinsma
Allowed aids during test: None

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. Consider the function $f(t)$ of period $T = 1$ that for $t \in [0,1]$ is defined as

$$f(t) = t - t^2 \quad \forall t \in [0,1].$$

   (a) Show that the Fourier coefficients equal $f_k = \dfrac{-1}{2k^2\pi^2}$ for all $k \neq 0$.
   (b) Determine its real Fourier series of $f(t)$.
   (c) Use the above to determine $\sum_{k=1}^{\infty} \frac{1}{k^4}$.
   (d) For which $t \in \mathbb{R}$ is the Fourier series at t the same as $f(t)$?

2. Let $h = f * g$. Prove from the integral definition of convolution that the convolution of $f(-t)$ and $g(-t)$ equals $h(-t)$.

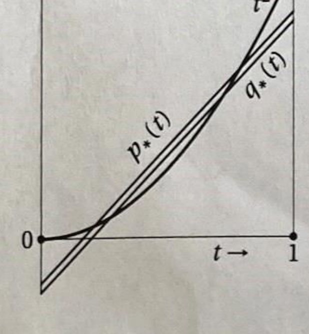
3. Which of the following six functions are equal to $\delta(t-1)$:

   (a) the derivative of $\mathbb{1}(2t-2)$
   (b) the derivative of $t\mathbb{1}(2t-2)$
   (c) $t\delta(2t-2)$
   (d) $2t^2\delta(2t-2)$
   (e) $\frac{1}{2}t\delta(2t-2)$
   (f) $t^2\big(\delta(t-1)+\delta(t)\big)$

4. What is the definition of a Hilbert space?

5. Consider $\mathscr{C}([0,1];\mathbb{R})$ with norm $\|f\| := \sqrt{\int_0^1 (t-t^2) f^2(t)\,\mathrm{d}t}$.

   (a) Formulate the corresponding inner product, and show that it is indeed an inner product on $\mathscr{C}([0,1];\mathbb{R})$.
   (b) Determine the best approximation of t^2 in the subspace $\mathscr{P}_1([0,1];\mathbb{R})$ of polynomials of degree 1 or less. Here we assume the above norm.
   (c) The figure below shows shows graphs of t^2 and its best approximation in $\mathscr{P}_1([0,1];\mathbb{R})$, as well as its best approximation in $\mathscr{P}_1([0,1];\mathbb{R})$ but then with respect to the *standard* 2-norm $\|f\|_2 := \sqrt{\int_0^1 f^2(t)\,\mathrm{d}t}$. Both best approximations are not very good near $t = 0$ and $t = 1$. Comparing the two norms, argue that p_* must correspond to the standard norm, and q_* to our funny norm $\|f\| := \sqrt{\int_0^1 (t-t^2) f^2(t)\,\mathrm{d}t}$.



problem:	1	2	3	4	5
points:	5+2+3+1	3	3	2	3+3+2

Test grade is $1 + p/3$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b)$	$f(t)$ continuous at $t = b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t = b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau), \ (\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \ (n \in \mathbb{Z})$	f_{k-n}