# Signals & Transforms (202001343) — TEST 2

| | |
|---|---|
| Date: | 10-06-2022 |
| Place: | TL 2275 |
| Time: | 08:45–10:15 (till 11:40 for students with special rights) |
| Course coordinator: | G. Meinsma |
| Allowed aids during test: | None |

**The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.**

1. The *Hann window* is defined as

$$f_{\text{hann}}(t) := \left(\tfrac{1}{2} + \tfrac{1}{2}\cos(\pi t)\right)\operatorname{rect}_2(t).$$

   (a) Determine the Fourier transform of the Hann window.

   (b) Determine $\int_{-\infty}^{\infty} |\hat{f}_{\text{hann}}(\omega)|^2 \, d\omega$.

2. Determine the convolution of $\operatorname{sinc}(t+1)$ and $\operatorname{sinc}(2t)$.

3. Given is the differential equation

$$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) - u^{(1)}(t). \tag{1}$$

   (a) Determine the frequency response of this differential equation.

   (b) Determine a solution $y(t)$ for the case that $u(t) = e^{2t}\,\mathbb{1}(-t)$.

   (c) Suppose that $u(t) = \mathbb{1}(t)$. Use Laplace transformation to determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 1$ and $y^{(1)}(0^-) = -2$.

4. Let $f : [0,\infty) \to \mathbb{R}$. What does it mean for this function to be *exponentially bounded*?

5. In the theory of Laplace there is a theorem about $\lim_{s\downarrow 0} sF(s)$, but this theorem does *not* apply to periodic functions. What can you say about $\lim_{s\downarrow 0} sF(s)$ for the case that $f(t)$ is periodic? [Hint: recall the result from chapter 3 that periodic functions $f(t)$ have a Fourier series $f(t) = \sum_{k=-\infty}^{\infty} f_k e^{ik\omega_0 t}$.]

| problem: | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| points: | 4+4 | 4 | 1+4+4 | 2 | 4 |

Test grade is $1 + 9p/p_{\max}$

| Property | Time domain | Freq. domain | Condition |
|---|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$ | |
| Duality | $\hat{f}(t)$ | $2\pi f(-\omega)$ | |
| Conjugation | $f^*(t)$ | $\hat{f}^*(-\omega)$ | |
| Time-scaling | $f(at)$ | $\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$ | $a \in \mathbb{R}, a \neq 0$ |
| Time-shift | $f(t-\tau)$ | $\hat{f}(\omega)e^{-i\omega\tau}$ | |
| Frequency-shift | $f(t)e^{i\omega_0 t}$ | $\hat{f}(\omega-\omega_0)$ | |
| Modulation Thm. | $f(t)\cos(\omega_0 t)$ | $\frac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$ | |
| Differentiation (time) | $f^{(1)}(t)$ | $(i\omega)\hat{f}(\omega)$ | $\lim_{t\to\pm\infty} f(t) = 0$ |
| Integration (time) | $\int_{-\infty}^{t} f(\tau)\,d\tau$ | $\frac{\hat{f}(\omega)}{i\omega}$ | $\hat{f}(0) = 0$ |
| Differentiation (freq.) | $-itf(t)$ | $\hat{f}'(\omega)$ | |

| $f(t)$ | $\hat{f}(\omega)$ | Condition |
|---|---|---|
| $\text{rect}_a(t)$ | $a\,\text{sinc}(a\omega/2)$ | $a > 0$ |
| $\text{trian}_a(t)$ | $a\,\text{sinc}^2(a\omega/2)$ | $a \in \mathbb{R},\ a > 0$ |
| $e^{-a\lvert t\rvert}$ | $\frac{2a}{a^2+\omega^2}$ | $\text{Re}(a) > 0$ |
| $\frac{t^n}{n!}e^{-at}\mathbb{1}(t)$ | $\frac{1}{(a+i\omega)^{n+1}}$ | $\text{Re}(a) > 0;\ n \in \mathbb{N}$ |
| $-\frac{t^n}{n!}e^{-at}\mathbb{1}(-t)$ | $\frac{1}{(a+i\omega)^{n+1}}$ | $\text{Re}(a) < 0;\ n \in \mathbb{N}$ |
| $e^{-(at)^2}$ | $\frac{\sqrt{\pi}}{\lvert a\rvert}e^{-(\omega/(2a))^2}$ | $a \in \mathbb{R},\ a \neq 0$ |
| $a\,\text{sinc}(at/2)$ | $2\pi\,\text{rect}_a(\omega)$ | $a \in \mathbb{R},\ a > 0$ |

| $f(t)$ | $\hat{f}(\omega)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $1$ | $2\pi\delta(\omega)$ |
| $\delta(t-b)$ | $e^{-i\omega b}$ |
| $e^{i\omega_0 t}$ | $2\pi\delta(\omega-\omega_0)$ |
| $\cos(\omega_0 t)$ | $\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$ |
| $\text{sgn}(t)$ | $\frac{2}{i\omega}$ |
| $\mathbb{1}(t)$ | $\frac{1}{i\omega} + \pi\delta(\omega)$ |

| Property | $f(t)$ | $F(s)$ |
|---|---|---|
| Linearity | $a_1 f_1(t) + a_2 f_2(t)$ | $a_1 F_1(s) + a_2 F_2(s)$ |
| Time-scaling | $f(at)$ | $\frac{1}{a}F(\frac{s}{a})$ (if $a > 0$) |
| Time-shift | $f(t-t_0)\mathbb{1}(t-t_0^-)$ | $F(s)e^{-st_0}$ (if $t_0 > 0$) |
| Shift in $s$-domain | $f(t)e^{s_0 t}$ | $F(s-s_0)$ |
| Differentiation ($t$) | $f^{(1)}(t)$ | $sF(s) - f(0^-)$ |
| | $f^{(2)}(t)$ | $s^2F(s) - sf(0^-) - f^{(1)}(0^-)$ |
| Integration ($t$) | $\int_{0^-}^{t} f(\tau)\,d\tau$ | $\frac{F(s)}{s}$ |
| Differentiation ($s$) | $-tf(t)$ | $F'(s)$ |

| $f(t),\ (t > 0^-)$ | $F(s)$ |
|---|---|
| $e^{at}$ | $\frac{1}{s-a}$ |
| $\frac{t^n}{n!}$ $(n \in \mathbb{N})$ | $\frac{1}{s^{n+1}}$ |
| $\frac{t^n}{n!}e^{at}$ $(n \in \mathbb{N})$ | $\frac{1}{(s-a)^{n+1}}$ |
| $\cos(bt)$ | $\frac{s}{s^2+b^2}$ |
| $\sin(bt)$ | $\frac{b}{s^2+b^2}$ |
| $e^{at}\cos(bt)$ | $\frac{s-a}{(s-a)^2+b^2}$ |
| $e^{at}\sin(bt)$ | $\frac{b}{(s-a)^2+b^2}$ |
| $\delta(t)$ | $1$ |