# Signals & Transforms (202001343) — TEST 1 RESIT

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| Date: | 13-06-2022 |
| Place: | Therm 2 |
| Time: | 13:45–15:15 (till 15:40 for students with special rights) |
| Course coordinator: | G. Meinsma |
| Allowed aids during test: | None |

**The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.**

1. Consider the function $f(t)$ of period $T = 2\pi$ which for $t \in [-\pi, \pi)$ is defined as

   $$f(t) = 2t\cos(t).$$

   (a) Determine the complex Fourier coefficients $f_k$.

   (b) Show that $\mathrm{Re}(f_k) = 0$ for all $k$.

   (c) Determine the real Fourier series of $f(t)$.

2. What is the *Gibbs phenomenon*? (A general idea suffices.)

3. Which of the following functions equal $\delta(t-1)$:

   (a) The derivative of $\mathbb{1}(2t-2)$

   (b) The derivative of $t\,\mathbb{1}(2t-2)$

   (c) $t\delta(2t-2)$

   (d) $2t^2\delta(2t-2)$

4. Let $u : \mathbb{R} \to \mathbb{R}$ be a piecewise smooth $T$-periodic function, and let $y$ be a $T$-periodic solution of the differential equation

   $$y^{(1)}(t) + y(t) = u(t).$$

   (a) Show that the power of this periodic function $y$ does not exceed the power of $u$ no matter which periodic $u$ we take.

   (b) Determine all periodic functions $u$ for which the power of the periodic solution $y$ equals that of $u$.

5. Use the integral defintion of convolution to determine the convolution of $f(t) = t\,\mathbb{1}(t-1)$ and $g(t) = t\,\mathbb{1}(t-1)$.

6. Consider $\mathscr{L}^2(\mathbb{R};\mathbb{C})$ with standard inner product $\langle g, h\rangle = \int_{-\infty}^{\infty} g(t)h^*(t)\,\mathrm{d}t$. In a later chapter (Chapter 4) it is shown that a certain mapping $\mathfrak{F} : \mathscr{L}^2(\mathbb{R};\mathbb{C}) \to \mathscr{L}^2(\mathbb{R};\mathbb{C})$ has the property that $\langle g, h\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)\hat{h}^*(\omega)\,\mathrm{d}\omega$, where $\hat{g} = \mathfrak{F}(g)$, $\hat{h} = \mathfrak{F}(h)$. Chapter 4 further claims that

   $$\mathfrak{F}\big(a\,\mathrm{sinc}(a(t-t_0)/2)\big) = 2\pi\,\mathrm{rect}_a(\omega)\mathrm{e}^{-\mathrm{i}\omega t_0}.$$

   (Here $t_0, a \in \mathbb{R}$ and $a > 0$.) Is the sequence of functions $v_k(t) := \mathrm{sinc}(\pi(t-k))$ ($k \in \mathbb{Z}$) an *orthonormal set* on $\mathscr{L}^2(\mathbb{R};\mathbb{C})$ in the standard inner product?

| problem: | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| points: | 5+1+3 | 2 | 2 | 2+2 | 5 | 5 |

Test grade is $1 + p/3$

| Property | | Condition |
|---|---|---|
| Sifting | $\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b)$ | $f(t)$ continuous at $t = b$ |
| - | $f(t)\delta(t-b) = f(b)\delta(t-b)$ | $f(t)$ continuous at $t = b$ |
| Convolution | $(f * \delta)(t) = f(t)$ | |
| Scaling | $\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$ | |
| - | $\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$ | $t \neq 0$ |

| Property | Time domain: $f(t)$ | Frequency domain: $f_k$ |
|---|---|---|
| Linearity | $\alpha f(t) + \beta g(t)$ | $\alpha f_k + \beta g_k$ |
| Time-shift | $f(t-\tau), \quad (\tau \in \mathbb{R})$ | $\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$ |
| Time-reversal | $f(-t)$ | $f_{-k}$ |
| Conjugation | $f^*(t)$ | $f^*_{-k}$ |
| Frequency-shift | $\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \quad (n \in \mathbb{Z})$ | $f_{k-n}$ |