Signals & Transforms (202001343) — TEST 2 RESIT

Date:	24-06-2022
Place:	CR 2K
Time:	13:45–15:15 (till 15:40 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	None

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as

$$f(t)=\begin{cases}-1 & -2<t<0\\ 1 & 0<t<2\\ 0 & \text{elsewhere}\end{cases}$$

   (a) Use the integral definition of the Fourier transform to determine the Fourier transform of $f(t)$.

   (b) Determine $\int_{-\infty}^{\infty}\frac{\sin^4(\omega)}{\omega^2}\,\mathrm{d}\omega$.

   (c) This function $f(t)$ is "zero on average" meaning that $\int_{-\infty}^{\infty}f(t)\,\mathrm{d}t=0$. Suppose g is absolutely integrable (and piecewise smooth). Show that $f*g$ is zero on average as well.

2. Let $a\in\mathbb{R}, a\neq 0$. The theory of Chapter 4 claims that the Fourier transform of $f(at)$ equals

$$\frac{1}{|a|}\hat{f}(\omega/a).$$

   Prove this result. (And, obviously, you are NOT allowed to use the tables now!)

3. Given is the differential equation

$$y^{(2)}(t)+4y^{(1)}(t)+4y(t)=u^{(1)}(t)+u(t). \tag{1}$$

   (a) Determine the frequency response of this differential equation.

   (b) Determine the impulse response of this differential equation.

   (c) Suppose that $u(t)=\mathrm{e}^t$. Use Laplace transformation to determine the solution $y(t)$ for $t>0$ of (1) for the case that $y(0^-)=0$ and $y^{(1)}(0^-)=2$.

4. Use the Laplace transform to determine all functions $y:[0,\infty)\to\mathbb{R}$ that satisfy

$$y(0)=0,\qquad \int_0^t y^{(1)}(\tau)y(t-\tau)\,\mathrm{d}\tau=t^4\quad\forall t>0.$$

problem:	1	2	3	4
points:	4+3+3	4	1+4+4	4

Test grade is $1+9p/p_{\max}$

Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\hat{f}\big(\frac{\omega}{a}\big)$	$a \in \mathbb{R}, a \neq 0$
Time-shift	$f(t-\tau)$	$\hat{f}(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}$	
Frequency-shift	$f(t)\mathrm{e}^{\mathrm{i}\omega_0 t}$	$\hat{f}(\omega-\omega_0)$	
Modulation Thm.	$f(t)\cos(\omega_0 t)$	$\frac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$	
Differentiation (time)	$f^{(1)}(t)$	$(\mathrm{i}\omega)\hat{f}(\omega)$	$\lim_{t\to\pm\infty} f(t) = 0$
Integration (time)	$\int_{-\infty}^{t} f(\tau)\,\mathrm{d}\tau$	$\frac{\hat{f}(\omega)}{\mathrm{i}\omega}$	$\hat{f}(0) = 0$
Differentiation (freq.)	$-\mathrm{i}tf(t)$	$\hat{f}'(\omega)$	

$f(t)$	$\hat{f}(\omega)$	Condition
$\mathrm{rect}_a(t)$	$a\,\mathrm{sinc}(a\omega/2)$	$a > 0$
$\mathrm{trian}_a(t)$	$a\,\mathrm{sinc}^2(a\omega/2)$	$a \in \mathbb{R},\ a > 0$
$\mathrm{e}^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$\mathrm{Re}(a) > 0$
$\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(t)$	$\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}(a) > 0;\ n \in \mathbb{N}$
$-\frac{t^n}{n!}\mathrm{e}^{-at}\,\mathbb{1}(-t)$	$\frac{1}{(a+\mathrm{i}\omega)^{n+1}}$	$\mathrm{Re}(a) < 0;\ n \in \mathbb{N}$
$\mathrm{e}^{-(at)^2}$	$\frac{\sqrt{\pi}}{\lvert a\rvert}\mathrm{e}^{-(\omega/(2a))^2}$	$a \in \mathbb{R},\ a \neq 0$
$a\,\mathrm{sinc}(at/2)$	$2\pi\,\mathrm{rect}_a(\omega)$	$a \in \mathbb{R},\ a > 0$

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t-b)$	$\mathrm{e}^{-\mathrm{i}\omega b}$
$\mathrm{e}^{\mathrm{i}\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
$\mathrm{sgn}(t)$	$\frac{2}{\mathrm{i}\omega}$
$\mathbb{1}(t)$	$\frac{1}{\mathrm{i}\omega} + \pi\delta(\omega)$

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time-scaling	$f(at)$	$\frac{1}{a}F\big(\frac{s}{a}\big)$ (if $a > 0$)
Time-shift	$f(t-t_0)\,\mathbb{1}(t-t_0^-)$	$F(s)\mathrm{e}^{-st_0}$ (if $t_0 > 0$)
Shift in s-domain	$f(t)\mathrm{e}^{s_0 t}$	$F(s-s_0)$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$
	$f^{(2)}(t)$	$s^2F(s) - sf(0^-) - f^{(1)}(0^-)$
Integration (t)	$\int_{0^-}^{t} f(\tau)\,\mathrm{d}\tau$	$\frac{F(s)}{s}$
Differentiation (s)	$-tf(t)$	$F'(s)$

$f(t),\ (t > 0^-)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
$\frac{t^n}{n!}$ ($n \in \mathbb{N}$)	$\frac{1}{s^{n+1}}$
$\frac{t^n}{n!}\mathrm{e}^{at}$ ($n \in \mathbb{N}$)	$\frac{1}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\mathrm{e}^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathrm{e}^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\delta(t)$	1