Signals & Transforms (202001343) — TEST 1RE

Date:	12-06-2023
Place:	CR-2M
Time:	08:45–10:15 (till 10:40 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	None

The solutions of the problems should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. Consider the periodic function $f(t)$ of period 2 which for $t \in [2,4)$ equals

   $$f(t) = t.$$

   (a) Determine the generalized derivative of $f(t)$ for all $t \in \mathbb{R}$.

   (b) Determine the complex Fourier coefficients f_k.

   (c) Determine the real Fourier series of $f(t)$.

   (d) Determine $\sum_{k=1}^{\infty} \frac{1}{k^2}$ using the Fourier coefficients.

2. Let $u : \mathbb{R} \to \mathbb{R}$ be a piecewise smooth T-periodic function, and let $y : \mathbb{R} \to \mathbb{R}$ be a T-periodic solution of the differential equation

   $$y^{(2)}(t) + 2y^{(1)}(t) + y(t) = u^{(1)}(t) - \alpha u(t).$$

   For which $\alpha \in \mathbb{R}$ is the average value of $y(t)$ equal to zero (independent of the choice of periodic u).

3. Determine the convolution of $f(t) = \delta(2t) + e^{2t}\mathbb{1}(-t)$ and $g(t) = e^{-t}\mathbb{1}(t-1)$. *You must use the integral definition of convolution, not Fourier or Laplace.*

4. Let $T > 0$, and suppose that $f(t)$ is a continuous function. Is it true that $f(t)$ has period T if-and-only-if $\int_a^{a+T} f(\tau)\,d\tau$ is the same for every $a \in \mathbb{R}$?

5. What is the definition of a 1-*norm* on the space of continuous functions $f : [0,2] \to \mathbb{C}$?

6. Let $\mathbb{X}$ be a real vector space with inner product, and let $\mathbb{V}$ a subspace of $\mathbb{X}$, and that $v_* \in \mathbb{V}$ is a best approximation in $\mathbb{V}$ of $x \in \mathbb{X}$. Prove that $x - v_* \perp \mathbb{V}$. *(You may only use elementary properties of inner products, and not the theorems in Chapter 1 that already claim that $x - v_* \perp \mathbb{V}$.)*

problem:	1	2	3	4	5	6
points:	2+4+2+3	1	6	3	1	5

Test grade is $1 + p/3$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b)$	$f(t)$ continuous at $t = b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t = b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau), \quad (\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \quad (n \in \mathbb{Z})$	f_{k-n}