Signals & Transforms (202001343) — TEST 2

Date: 28-03-2023
Place: NH-115
Time: 13:45–15:15 (till 15:40 for students with special rights)
Course coordinator: G. Meinsma
Allowed aids during test: None

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. *The Gibbs phenomenon of the Fourier transform.*

   (a) Let $N \gg 0$. Determine the function $g_N(t)$ such that

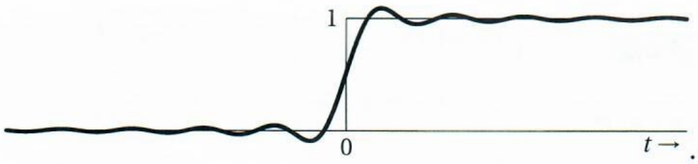
   $$(f * g_N)(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}(\omega)\,\mathrm{rect}_{2N}(\omega).$$

   (b) Interpretation: argue that $(f * g_N)(t)$ is an approximation of $f(t)$ obtained by removing all "large" frequencies from $f(t)$.

   (c) Let $T > 0$. Show that for $f(t) = \mathbb{1}(t) - \mathbb{1}(t-T)$ we have

   $$(f * g_N)(t) = \int_{t-T}^{t} g_N(\tau)\,\mathrm{d}\tau.$$

   (d) In the limit $T \to \infty$ the above $f(t)$ becomes the unit step, $f(t) = \mathbb{1}(t)$, and $(f * g_N)(t) = \int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$. It can be shown that $\int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$ looks like

   Let $N > 0$. Show that $\max_{t\in\mathbb{R}} \int_{-\infty}^{t} g_N(\tau)\,\mathrm{d}\tau$ does not depend on N.

2. Proof the conjugation property of Fourier transformation: $f^*(t) \overset{\mathfrak{F}}{\longleftrightarrow} \hat{f}^*(-\omega)$. [Of course you may NOT use the tables in this case.]

3. Determine the convolution of $\sin(t)\,\mathbb{1}(t)$ and $g(t) = (1+\mathrm{e}^{-t})\,\mathbb{1}(t)$ using the Laplace transform.

4. Given is the differential equation

   $$y^{(2)}(t) + 3y^{(1)}(t) + 2y(t) = u^{(2)}(t) - u^{(1)}(t). \qquad (1)$$

   (a) Determine the frequency response of this differential equation.

   (b) Determine a solution $y(t)$ for the case that $u(t) = \mathrm{e}^{2t}\,\mathbb{1}(-t)$.

   (c) Suppose that $u(t) = \mathbb{1}(t)$. Use Laplace transformation to determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 1$ and $y^{(1)}(0^-) = -2$.

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problem:	1	2	3	4
points:	3+1+3+3	2	5	1+4+5

Test grade is $1 + 9p/p_{\max}$

Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$	$a \in \mathbb{R}, a \neq 0$
Time-shift	$f(t-\tau)$	$\hat{f}(\omega)e^{-i\omega\tau}$	
Frequency-shift	$f(t)e^{i\omega_0 t}$	$\hat{f}(\omega - \omega_0)$	
Modulation Thm.	$f(t)\cos(\omega_0 t)$	$\frac{\hat{f}(\omega-\omega_0)+\hat{f}(\omega+\omega_0)}{2}$	
Differentiation (time)	$f^{(1)}(t)$	$(i\omega)\hat{f}(\omega)$	$\lim_{t\to\pm\infty} f(t) = 0$
Integration (time)	$\int_{-\infty}^{t} f(\tau)\,d\tau$	$\frac{\hat{f}(\omega)}{i\omega}$	$\hat{f}(0) = 0$
Differentiation (freq.)	$-itf(t)$	$\hat{f}'(\omega)$	

$f(t)$	$\hat{f}(\omega)$	Condition
$\mathrm{rect}_a(t)$	$a\,\mathrm{sinc}(a\omega/2)$	$a > 0$
$\mathrm{trian}_a(t)$	$a\,\mathrm{sinc}^2(a\omega/2)$	$a \in \mathbb{R},\ a > 0$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$\mathrm{Re}(a) > 0$
$\frac{t^n}{n!}e^{-at}\mathbb{1}(t)$	$\frac{1}{(a+i\omega)^{n+1}}$	$\mathrm{Re}(a) > 0;\ n \in \mathbb{N}$
$-\frac{t^n}{n!}e^{-at}\mathbb{1}(-t)$	$\frac{1}{(a+i\omega)^{n+1}}$	$\mathrm{Re}(a) < 0;\ n \in \mathbb{N}$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{\lvert a\rvert}e^{-(\omega/(2a))^2}$	$a \in \mathbb{R},\ a \neq 0$
$a\,\mathrm{sinc}(at/2)$	$2\pi\,\mathrm{rect}_a(\omega)$	$a \in \mathbb{R},\ a > 0$

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t-b)$	$e^{-i\omega b}$
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$
$\mathrm{sgn}(t)$	$\frac{2}{i\omega}$
$\mathbb{1}(t)$	$\frac{1}{i\omega} + \pi\delta(\omega)$

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time-scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$ (if $a > 0$)
Time-shift	$f(t-t_0)\mathbb{1}(t-t_0^-)$	$F(s)e^{-st_0}$ (if $t_0 > 0$)
Shift in s-domain	$f(t)e^{s_0 t}$	$F(s-s_0)$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$
	$f^{(2)}(t)$	$s^2F(s) - sf(0^-) - f^{(1)}(0^-)$
Integration (t)	$\int_{0^-}^{t} f(\tau)\,d\tau$	$\frac{F(s)}{s}$
Differentiation (s)	$-tf(t)$	$F'(s)$

$f(t),\ (t > 0^-)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
$\frac{t^n}{n!}$ ($n \in \mathbb{N}$)	$\frac{1}{s^{n+1}}$
$\frac{t^n}{n!}e^{at}$ ($n \in \mathbb{N}$)	$\frac{1}{(s-a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$\delta(t)$	1