Signals & Transforms (202001343) — TEST 1

Date:	13-03-2023
Place:	HT-1100
Time:	08:45–10:15 (till 10:40 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	None

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer! You are not allowed to use a calculator.

1. Let $\alpha \in \mathbb{R}$ and consider the complex function $f(t)$ of period 2π which for $t \in [0, 2\pi)$ equals

$$f(t) = \begin{cases} e^{i\alpha t} & \text{if } t \in [0, \pi), \\ 0 & \text{if } t \in [\pi, 2\pi). \end{cases}$$

   (a) Determine the generalized derivative of $f(t)$ for all $t \in \mathbb{R}$.

   (b) Determine the complex Fourier coefficients f_k.

   (c) Determine all $\alpha \in \mathbb{R}$ for which infinitely many Fourier coefficients are equal to zero.

   (d) Determine all $t \in \mathbb{R}$ for which the Fourier series at t equals $f(t)$.

2. Let $u : \mathbb{R} \to \mathbb{R}$ be a piecewise smooth T-periodic function, and let $y : \mathbb{R} \to \mathbb{R}$ be a T-periodic solution of the differential equation

$$y^{(1)}(t) + 3y(t) = u^{(1)}(t) + 2u(t).$$

   (a) Express the Fourier coefficients y_k of y in terms of the Fourier coefficients u_k of u.

   (b) Show that the power of y is less than or equal to the power of u.

3. Determine the convolution of $f(t) = \text{rect}_2(t-1)$ and $g(t) = e^t \mathbb{1}(-t)$.

4. let $A > 0, B > 0, \phi_1, \phi_2 \in \mathbb{R}$. Show using Euler's formula that

$$A\cos(t + \phi_1) + B\cos(t + \phi_2) = C\cos(t + \phi_3)$$

   for some $C \geq 0, \phi_3 \in \mathbb{R}$.

5. What is the definition of a *Banach space.*

6. Let $\mathbb{X}$ be a complex Hilbert space, and suppose $e_1, e_2, \ldots$ is an infinite orthonormal sequence in $\mathbb{X}$. Let $a_k \in \mathbb{C}$ for $k \in \mathbb{N}$. Prove that $\sum_{k=1}^{\infty} a_k e_k$ converges in $\mathbb{X}$ if-and-only-if $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

problem:	1	2	3	4	5	6
points:	2+4+2+1	1+2	4	4	2	4

Test grade is $1 + p/3$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b)$	$f(t)$ continuous at $t=b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t=b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a \rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau), \quad (\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0 \tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \quad (n \in \mathbb{Z})$	f_{k-n}