Signals & Transforms (202001343) — TEST 1

Date:	10-03-2024
Place:	SP 1
Time:	08:45–10:15 (till 10:40 for students with special facilities)
Course coordinator:	G. Meinsma
Allowed aids during test:	None

The solutions of the exercises should be clearly formulated. You are not allowed to use a calculator.

1. Consider the periodic function $f(t)$ with period $T = 4$ which for $t \in [-1,3)$ equals
$$f(t) = \begin{cases} 2 & t \in [-1,1) \\ -1 & t \in [1,3) \end{cases}.$$
   (a) Determine the Fourier coefficients f_k of $f(t)$ and show that they are real numbers.
   (b) For which $t \in \mathbb{R}$ is the Fourier series of $f(t)$ equal to $f(t)$?
   (c) Determine the real Fourier series of $f(t)$.
2. Let $f(t)$ be a T-periodic function, and let $g(t) = f(3t)$. Express the Fourier coefficients g_k of $g(t)$ in terms of the Fourier coefficients f_k of $f(t)$.
3. Determine the generalized derivative of $\mathrm{e}^{2t}\,\mathbb{1}(-t) + \mathbb{1}(t^2 - 9)$ and make your final answer as simple as possible.
4. Determine the convolution of $f(t) = \operatorname{rect}_2(t-1)$ and $g(t) = \mathrm{e}^t\,\mathbb{1}(-t)$.
5. Determine $A \geq 0$ and a $\phi \in \mathbb{R}$ such that $\cos(t) - \sin(t) = A\cos(t + \phi)$.
6. Let $\mathbb{X}$ be an inner product space and suppose $(e_k)_{k\in\mathbb{N}}$ an orthonormal sequence in $\mathbb{X}$. Let $\alpha_k \in \mathbb{R}$.
   (a) If $\mathbb{X}$ is complete (i.e. a Hilbert space), show that $\sum_{k\in\mathbb{N}} \alpha_k e_k$ is in $\mathbb{X}$ if-and-only-if $\sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$.
   (b) Give an example of an inner product space $\mathbb{X}$ for which $\sum_{k\in\mathbb{N}} |\alpha_k|^2 < \infty$ yet $\sum_{k\in\mathbb{N}} \alpha_k e_k$ is not in $\mathbb{X}$.

problem:	1	2	3	4	5	6
points:	6+1+2	2	2	5	2	5+2

Test grade is $1 + p/3$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\, \mathrm{d}t = f(b)$	$f(t)$ continuous at $t = b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t = b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a \rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\, \mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau), \quad (\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \quad (n \in \mathbb{Z})$	f_{k-n}