

Signals & Transforms (202001343) — TEST 2

Date: 02-04-2025
 Place: OH-211
 Time: 13:45–15:15 (till 15:40 for students with special facilities)
 Course coordinator: G. Meinsma
 Allowed aids during test: None

The solutions of the exercises should be clearly formulated. Moreover, in all cases you should motivate your answer, unless explicitly stated otherwise! You are not allowed to use a calculator.

1. Sketch the graph of $|\hat{f}(\omega)|$ for the case that $f(t) = (1 + \cos(2t) + 2\sin(3t))e^{-t^2/10}$.
(A rough sketch suffices; mathematical formulas are not required.)
2. Consider the function $f(t) = t^2 \text{rect}_2(t - 1)$. Use the integral-definition of the Fourier transform to determine the Fourier transform of $f(t)$.
3. Express the Fourier transform of $\text{Re}(f(2t))$ in terms of the Fourier transform of $f(t)$.
4. Suppose we know that $f(t)$ has bandwidth 10. For which sampling periods $T_s > 0$ can we reconstruct $f(t)$ for all $t \in \mathbb{R}$ based on samples $f(kT_s), k \in \mathbb{Z}$?
5. Determine the Laplace transform of $\delta(t + 1) + \delta(t - 1) + t \cos(2t)$.
6. Given is the differential equation

$$y^{(2)}(t) - y(t) = 2u^{(1)}(t) - u(t). \quad (1)$$

- (a) Determine the frequency response of this differential equation.
- (b) Determine a solution $y(t)$ for the case that $u(t) = e^t \mathbb{1}(-t)$.
- (c) Suppose $u(t) = e^{2t} \mathbb{1}(t)$. Use Laplace transformation to determine the solution $y(t)$ for $t > 0$ of (1) for the case that $y(0^-) = 1$ and $y^{(1)}(0^-) = 2$.

problem:	1	2	3	4	5	6
points:	2	6	3	2	3	1+4+6

Test grade is $1 + 9p/p_{\max}$

Property	Time domain	Freq. domain	Condition
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 \hat{f}_1(\omega) + a_2 \hat{f}_2(\omega)$	$a \in \mathbb{R}, a \neq 0$
Duality	$\hat{f}(t)$	$2\pi f(-\omega)$	
Conjugation	$f^*(t)$	$\hat{f}^*(-\omega)$	
Time-scaling	$f(at)$	$\frac{1}{ a } \hat{f}\left(\frac{\omega}{a}\right)$	
Time-shift	$f(t - \tau)$	$\hat{f}(\omega) e^{-i\omega\tau}$	
Frequency-shift	$f(t) e^{i\omega_0 t}$	$\hat{f}(\omega - \omega_0)$	
Modulation Thm.	$f(t) \cos(\omega_0 t)$	$\frac{\hat{f}(\omega - \omega_0) + \hat{f}(\omega + \omega_0)}{2}$	
Differentiation (time)	$f^{(1)}(t)$	$(i\omega) \hat{f}(\omega)$	$\lim_{t \rightarrow \pm\infty} f(t) = 0$ $\hat{f}(0) = 0$
Integration (time)	$\int_{-\infty}^t f(\tau) d\tau$	$\frac{\hat{f}(\omega)}{i\omega}$	
Differentiation (freq.)	$-it f(t)$	$\hat{f}'(\omega)$	

$f(t)$	$\hat{f}(\omega)$	Condition
$\text{rect}_a(t)$	$a \text{sinc}(a\omega/2)$	$a > 0$
$\text{trian}_a(t)$	$a \text{sinc}^2(a\omega/2)$	$a \in \mathbb{R}, a > 0$
$e^{-a t }$	$\frac{2a}{a^2 + \omega^2}$	$\text{Re}(a) > 0$
$\frac{t^n}{n!} e^{-at} \mathbb{1}(t)$	$\frac{1}{(a + i\omega)^{n+1}}$	$\text{Re}(a) > 0; n \in \mathbb{N}$
$-\frac{t^n}{n!} e^{-at} \mathbb{1}(-t)$	$\frac{1}{(a + i\omega)^{n+1}}$	$\text{Re}(a) < 0; n \in \mathbb{N}$
$e^{-(at)^2}$	$\frac{\sqrt{\pi}}{ a } e^{-(\omega/(2a))^2}$	$a \in \mathbb{R}, a \neq 0$
$a \text{sinc}(at/2)$	$2\pi \text{rect}_a(\omega)$	$a \in \mathbb{R}, a > 0$

$f(t)$	$\hat{f}(\omega)$
$\delta(t)$	1
1	$2\pi\delta(\omega)$
$\delta(t - b)$	$e^{-i\omega b}$
$e^{i\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 t)$	$\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0))$
$\text{sgn}(t)$	$\frac{2}{i\omega}$
$\mathbb{1}(t)$	$\frac{1}{i\omega} + \pi\delta(\omega)$

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Time-scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$ (if $a > 0$)
Time-shift	$f(t - t_0) \mathbb{1}(t - t_0^-)$	$F(s) e^{-st_0}$ (if $t_0 > 0$)
Shift in s -domain	$f(t) e^{s_0 t}$	$F(s - s_0)$
Differentiation (t)	$f^{(1)}(t)$	$sF(s) - f(0^-)$
	$f^{(2)}(t)$	$s^2 F(s) - s f(0^-) - f^{(1)}(0^-)$
Integration (t)	$\int_{0^-}^t f(\tau) d\tau$	$\frac{F(s)}{s}$
Differentiation (s)	$-t f(t)$	$F'(s)$

$f(t), (t > 0^-)$	$F(s)$
e^{at}	$\frac{1}{s - a}$
$\frac{t^n}{n!} (n \in \mathbb{N})$	$\frac{1}{s^{n+1}}$
$\frac{t^n}{n!} e^{at} (n \in \mathbb{N})$	$\frac{1}{(s - a)^{n+1}}$
$\cos(bt)$	$\frac{s}{s^2 + b^2}$
$\sin(bt)$	$\frac{b}{s^2 + b^2}$
$e^{at} \cos(bt)$	$\frac{s - a}{(s - a)^2 + b^2}$
$e^{at} \sin(bt)$	$\frac{b}{(s - a)^2 + b^2}$
$\delta(t)$	1