Signals & Transforms (202001343) — TEST 1 RE

Date:	09-04-2025
Place:	TL-3130
Time:	13:45–15:15 (till 15:40 for students with special facilities)
Course coordinator:	G. Meinsma
Allowed aids during test:	None

The solutions of the exercises should be clearly formulated. You are not allowed to use a calculator.

1. Consider the periodic function $f\colon \mathbb{R} \to \mathbb{C}$ with period π which for $t \in [0, \pi)$ equals

   $$f(t) = \mathrm{e}^{\mathrm{i}t}.$$

   (a) Determine the complexe Fourier coefficients f_k of $f(t)$.

   (b) Determine the value of the Fourier series at $t = 0$.

   (c) Determine the real Fourier series of $\operatorname{Re}(f(t))$.

2. Determine the generalized derivative of $\mathbb{1}(-t+2)\cos(t - \mathbb{1}(t))$ and make your final answer as simple as possible.

3. Determine the convolution of $f(t) = \mathrm{e}^{t}\,\mathbb{1}(-t)$ and $g(t) = \mathrm{e}^{2t}\,\mathbb{1}(-t+1)$. *You must use the integral definition of convolution, not Fourier or Laplace.*

4. Formulate the *reverse triangle inequality* for norms.

5. The lecture notes argue that the sequence $(\frac{1}{\sqrt{2}}\mathrm{e}^{\mathrm{i}k\pi t})_{k\in\mathbb{Z}}$ is a complete orthonormal sequence for $\mathscr{L}^2([-1,1];\mathbb{C})$.

   (a) Show that every $f \in \mathscr{L}^2([0,1];\mathbb{R})$ can be written as

   $$f(t) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\pi t), \qquad c_k \in \mathbb{R}.$$

   [Hint: given $f \in \mathscr{L}^2([0,1];\mathbb{R})$ define $g\colon (-1,1) \to \mathbb{R}$ as $g(t) = f(|t|)$.]

   (b) For each $k \in \mathbb{N} = \{1,2,\ldots\}$ determine $\|\cos(k\pi t)\|$ where $\|\cdot\|$ is the norm on $\mathscr{L}^2([0,1];\mathbb{R})$.

problem:	1	2	3	4	5
points:	4+2+4	3	5	2	4+3

Test grade is $1 + p/3$

Property		Condition
Sifting	$\int_{-\infty}^{\infty} \delta(t-b) f(t)\,\mathrm{d}t = f(b)$	$f(t)$ continuous at $t=b$
-	$f(t)\delta(t-b) = f(b)\delta(t-b)$	$f(t)$ continuous at $t=b$
Convolution	$(f * \delta)(t) = f(t)$	
Scaling	$\delta(at-b) = \frac{1}{\lvert a\rvert}\delta(t-\frac{b}{a})$	
-	$\int_{-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau = \mathbb{1}(t)$	$t \neq 0$

Property	Time domain: $f(t)$	Frequency domain: f_k
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha f_k + \beta g_k$
Time-shift	$f(t-\tau), \ (\tau \in \mathbb{R})$	$\mathrm{e}^{-\mathrm{i}k\omega_0\tau} f_k$
Time-reversal	$f(-t)$	f_{-k}
Conjugation	$f^*(t)$	f^*_{-k}
Frequency-shift	$\mathrm{e}^{\mathrm{i}n\omega_0 t} f(t), \ (n \in \mathbb{Z})$	f_{k-n}