(Class) Test-2: Analysis II
Statistics and Analysis (201800139)

23-October-2018, 08:45 – 10:15 Total Points : 22

All answers must be motivated.
Approach to a solution is equally (if not more) important.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. (a.) Give the definitions of a metric space and a convergent sequence in it. [3]

   (b.) Consider a convergent sequence x_n in a metric space X. Suppose $E \subset X$ is a closed set. Show that if $x_n \in E$ for all $n \in \mathbb{N}$, then the limit is also in E. [3]

   (c.) Consider the open and closed balls (with radius $r > 0$) around a in a metric space (X, ρ):

$$B_r(a) = \{x \in X : \rho(x, a) < r\} \quad \text{and} \quad C_r(a) = \{x \in X : \rho(x, a) \leq r\}.$$

   Show that the closure, $\overline{B_r(a)}$, of the open ball need not coincide with the corresponding closed ball $C_r(a)$. [2]

2. (a.) Let X and Y be two metric spaces and $f : X \to Y$ be a continuous function. Suppose $E \subset X$ is a connected set. Show that the forward image $f(E)$ is connected in Y. [3]

   [You may use relationships regarding inverse images without proof.]

   (b.) Show that the inverse image of a connected set need not be connected. [1]

3. (a.) Give the definition of differentiability of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, at a point $\mathbf{a} \in \mathbb{R}^n$. [1]

   (b.) Prove the *chain rule* for functions of more than one variables: [4]

   Suppose $g : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a}$ and $f : \mathbb{R}^m \to \mathbb{R}^p$ is differentiable at $g(\mathbf{a})$. Then $f \circ g$ is differentiable at $\mathbf{a}$ and

$$D(f \circ g)(\mathbf{a}) = Df(g(\mathbf{a}))\, Dg(\mathbf{a}).$$

4. (a.) State the implicit function theorem. [2]

   (b.) Consider the following relations:

$$\begin{aligned} u^2 + x\,v + y &= 0 \\ y\,u + v^3 + x^2 &= 0. \end{aligned}$$

   Do these relationships allow us to consider x and y to be (proper) functions of u and v defined on a (non-empty) neighbourhood of the point $(u_0, v_0) = (0, -1)$? [3]

Grade:	$\dfrac{\text{score on test}}{22} \times 9 + 1$ (rounded off to two decimal places)