~~Sample~~ Make-up Exam: Analysis II
Statistics and Analysis (201800139)

08-November-2018, 08:45 – 11:45 Total Points : 35

All answers must be motivated.
Approach to a solution is equally (if not more) important.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. (a.) Define the absolute convergence of a series of real numbers, using ϵ-δ-N arguments/ language. [2]

   (b.) Suppose the series $\sum_{k=1}^{\infty} a_k$ converges. Prove that $a_k \to 0$ as $k \to \infty$. [2]

   (c.) Determine the convergence/divergence of the following series. In case a series converges, determine whether it converges absolutely. [2+2]

   (i.) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{k^k}$

   (ii.) $\displaystyle\sum_{k=1}^{\infty} (-1)^k\, k \sin\left(\frac{1}{k}\right)$

2. (a.) Consider the function $f(x) = (x+4)^{-1}$. Prove that the Taylor series of f around $x_0 = 0$ is given by
$$\sum_{k=0}^{\infty} (-1)^k \frac{1}{4^{k+1}} x^k.$$
[3]

   (b.) Determine the radius of converges and the interval of convergence for the power series from (a). [2]

3. Let $X := \mathcal{C}([0,1])$ denote the set of continuous functions over the interval $[0,1]$.

(a.) Consider the map $\rho : X \times X \to \mathbb{R}$, defined by

$$\rho(f,g) := \sup_{x \in [0,1]} |f(x) - g(x)| \quad \text{for all } f, g \in X.$$

Prove that (X, ρ) is a metric space. [3]

(b.) Prove that a sequence f_n in X converges with respect to the metric ρ if and only if the sequence of real-valued functions $f_n : [0,1] \to \mathbb{R}$ converges uniformly on $[0,1]$. [3] 1

(c.) Prove that the set

$$E := \{f \in X : \exists x \in [0,1] \text{ with } f(x) > 0\}$$

is an open subset of X. [4] 2

4. Let $n \geq 1$ be a positive integer and consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = |x|^n + |y|^n.$$

(a.) Show that f is differentiable on $\mathbb{R}^2 \setminus \{0\}$ and compute its gradient vector. [2] 1

(b.) Show that f is differentiable at $(x,y) = (0,0)$ if and only if $n \geq 2$. [2]

5. (a.) State the Inverse Function Theorem. [2]

(b.) Consider the map $f_t : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f_t(x,y) = \begin{pmatrix} x^3 + t(x + x^4 y^7) \\ y^3 + t(y + x^3 y^2) \end{pmatrix}$$

depending on the real parameter $t \in \mathbb{R}$. Prove that, for any choice of t, there exists an open neighbourhood V around $(0,0)$ such that the inverse f_t^{-1} exists on the set V.

[4] 1 ?

(c.) For which values of t is f_t^{-1} differentiable at $(0,0)$? [2]

Grade: $\dfrac{\text{score on test}}{35} \times 9 + 1$ (rounded off to one decimal place)