(Class) Test-2: Analysis II

Statistics and Analysis (201800139)

22-October-2019, 08:45 – 10:15 Total Points : 21

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. (a.) Give the definition of a metric space and an open set in it. [2]

   (b.) Let X be a metric space and $A \subset X$ be such that there exists a sequence $\{x_n\} \subset A$ which converges to $x_0 \in A^c := X \setminus A$. Are the following statements true or false? [3]

   (i.) A is dense in X. (ii.) $\{x_n\}$ is a Cauchy sequence (in X).

   (iii.) $x_0 \in \overline{A}$, the closure of A.

2. (a.) Let $f : X \to Y$ be a function between two metric spaces. Show that f is continuous on X if and only if the inverse image of every open set is open. [3]

   (b.) Prove that, in a metric space, any closed subset of a compact set is compact. [3]

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} y\,\frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

   (a.) Show that the first order partial derivatives of f exist everywhere on $\mathbb{R}^2$. [2]

   (b.) Determine whether f is differentiable on $\mathbb{R}^2$. [3]

4. (a.) State the implicit function theorem. [2]

   (b.) Consider the following relations:

$$\begin{aligned} u\,x^3 - v\,y &= 2 \\ u^2\,y^3 + v\,x &= 0. \end{aligned}$$

   Do these relationships allow us to consider u and v to be (proper) functions of x and y defined on a (non-empty) neighbourhood of the point $(x_0, y_0) = (1, -1)$? If such functions do exist, what sort of properties (continuity, differentiability, etc.) do they have? [3]

Grade: $\dfrac{\text{score on test}}{21} \times 9 + 1$ (rounded off to two decimal places)