Make-up Exam: Analysis II
Statistics and Analysis (201800139)

7-November-2019, 08:45 – 11:45

Total Points : 34

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. Consider the series: $\displaystyle\sum_{k=1}^{\infty}(-1)^k \frac{2k+3}{(k+1)(k+2)}$.

   (a.) Show that the series converges. Find also its value. [2+1]

   [Hint: Splitting the fraction can be helpful, especially to find the value.]

   (b.) Determine whether the series is absolutely convergent. [1]

2. Let $p \in \mathbb{R}$ and $p \geq 0$. Determine the necessary and sufficient condition on p such that the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\,(\ln k)^p}$ converges. [3]

3. (a.) Give the definition of uniform convergence of a sequence of real-valued functions, using ϵ-δ-N arguments/language. [1]

   (b.) Consider the sequence of functions, given by (for $n \geq 1$): $g_n(x) = e^{-(x^2+\frac{1}{n^2})}$, $x \in \mathbb{R}$. From the definition, show that g_n converges uniformly on $\mathbb{R}$. [2]

   (c.) Let $E \subseteq \mathbb{R}$ be a non-empty set and $f_n : E \to \mathbb{R}$ be a sequence of functions. Suppose f_n converges to some real-valued function, f, uniformly on E. Suppose that $x_0 \in E$ and for each $n \in \mathbb{N}$, f_n is continuous at x_0. Show that f is continuous at x_0. [3]

4. Consider the real-valued function f given as a series: $\displaystyle f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2}$, $x \in \mathbb{R}$.

   Show that $\displaystyle\int_0^{\pi/2} f(x)\,dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}$. [3]

5. (a.) Define the *completeness* property of a metric space and give an example of an incomplete metric space. [1+1]

   (b.) Let (X,ρ) be a metric space and $E \subseteq X$ be a closed set. Suppose $x_n \in E$, for each $n \in \mathbb{N}$ and the sequence $\{x_n\}$ converges (in X). Show that the limit $x := \lim_{n\to\infty} x_n \in E$. [3]

6. Let X be a metric space and $E \subseteq X$.

X (a.) Give the definition of the interior, E^0, and the boundary, ∂E, of E. [1+1]

(b.) Show that $x \notin E^0$ if and only if $B_r(x) \cap E^c \neq \emptyset$ for every $r > 0$. [3]

[This result has been used in the proof of: $\partial E = \overline{E} \setminus E^0$. Thus, your proof should not use the latter.]

7. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & x \neq 0, \ y \in \mathbb{R} \\ 0 & x = 0, \ y \in \mathbb{R}. \end{cases}$

Determine whether f is differentiable at the point $(0, 0)$. [4]

8. For this problem, assume that $\mathbb{R}^n$ consists of column-vectors. Now, suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ with the total derivative $Df(\mathbf{a})$, which is a row-vector. Suppose further that $f(\mathbf{a} + \mathbf{h}) \neq 0$, for all $\mathbf{h} \in \mathbb{R}^n$ with sufficiently small $\|\mathbf{h}\|$.

Clearly, then, $g := \dfrac{1}{f}$ is well defined in a neighbourhood of $\mathbf{a}$. In the following, you will show that g is also differentiable at $\mathbf{a}$ with

$$Dg(\mathbf{a}) = -\frac{Df(\mathbf{a})}{[f(\mathbf{a})]^2}.$$

Towards this end, it is important to analyze the difference $\frac{1}{f(\mathbf{a}+\mathbf{h})} - \frac{1}{f(\mathbf{a})}$. With a simple algebraic manipulation it can be shown that if $f(\mathbf{a} + \mathbf{h}) \neq 0$, then

$$\frac{1}{f(\mathbf{a}+\mathbf{h})} - \frac{1}{f(\mathbf{a})} + \frac{(Df(\mathbf{a}))\,\mathbf{h}}{[f(\mathbf{a})]^2}$$
$$= \frac{f(\mathbf{a}) - f(\mathbf{a}+\mathbf{h}) + (Df(\mathbf{a}))\,\mathbf{h}}{f(\mathbf{a})f(\mathbf{a}+\mathbf{h})} + \frac{[f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})]\,(Df(\mathbf{a}))\,\mathbf{h}}{[f(\mathbf{a})]^2\, f(\mathbf{a}+\mathbf{h})}$$

(a.) Argue that $\dfrac{(Df(\mathbf{a}))\,\mathbf{h}}{\|\mathbf{h}\|}$ is bounded for all $\mathbf{h} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. [1]

(b.) Show that g is differentiable at $\mathbf{a}$ with the total derivative $Dg(\mathbf{a})$ as given above. [3]

Grade:	$\dfrac{\text{score on test}}{34} \times 9 + 1$ (rounded off to one decimal place)