(Class) Test-2: Analysis II
Statistics and Analysis (202001350)

20-October-2020, 10:00 – 11:30 Total Points : 20

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. (a) Give the definition of a metric space and closed set in it. [2]

   (b) Let X be a metric space with metric ρ. Consider the ball with center a and radius r in X, i.e., $B_r(a) = \{x \in X \mid \rho(x, a) < r\}$.

   i. Show that the closure of $B_r(a)$ is contained in $\{x \in X \mid \rho(x, a) \leq r\}$. [2]

   ii. Give an example of a metric space for which $\{x \in X \mid \rho(x, a) \leq r\}$ is unequal to the closure of $B_r(a)$ for some a and $r > 0$. [2]

2. (a) Prove that, in a metric space, any compact set is bounded. [3]

   (b) Let $f : X \to Y$ be a continuous function between two metric spaces. Show that $f(K)$ is compact in Y when K is compact in X. [3]
   You may use that $f^{-1}(O)$ is open in X whenever O is open in Y

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x, y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & (x, y) \neq (0, 0), \\ 1 & (x, y) = (0, 0). \end{cases}$$

   (a) Show that f differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$. [2]

   (b) Determine whether f is differentiable in $(0, 0)$. [3]

4. Does there exist an open set $W \subset \mathbb{R}^2$ containing the point $(1, -1)$ and two functions $g_1, g_2 : W \mapsto \mathbb{R}$ continuously differentiable on W such that

$$g_1(1, -1) = 1, \qquad g_2(1, -1) = 1,$$

and

$$\begin{aligned} g_1(x, y)^2 x^3 - g_2(x, y)^2 y &= 2 \\ g_1(x, y)^3 e^x + \frac{e^{-y}}{g_2(x, y)} &= 2e \end{aligned}$$

for all $(x, y) \in W$? [3]

Grade:	$\frac{\text{score on test}}{20} \times 9 + 1$ (rounded off to one decimal place)