Make-up Exam: Analysis II
Statistics and Analysis (201800139)

November, 5, 2020, 09:00 - 12:00 Total Points : 38

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. Let $p \in \mathbb{R}$. Furthermore, let the sequence $\{a_k,\ k \in \mathbb{N}\}$ be such that

$$\sum_{k=1}^{\infty} |a_k| < \infty. \tag{1}$$

   (a) Prove that the series [2]

$$\sum_{k=1}^{\infty} \frac{a_k}{k^p}$$

   converges when $p \geq 0$.

   (b) Given $p < 0$. Construct a sequence $\{a_k,\ k \in \mathbb{N}\}$ satisfying (1) such that the series [2]

$$\sum_{k=1}^{\infty} \frac{a_k}{k^p}$$

   diverges.

2. Find the radius of convergence and the convergence interval of the power series: [4]

$$\sum_{k=1}^{\infty} (-1)^k \frac{2k+3}{(k^2+1)} x^k.$$

3. (a) Give the definition of uniform convergence of a sequence of real-valued functions, using ϵ-δ-N arguments/language. [1]

   (b) Give the definition of convergence of a sequence of real-valued functions, using ϵ-δ-N arguments/language. [1]

   (c) Define the following sequence of functions $f_n : (-1, 1) \mapsto \mathbb{R}$

$$f_n(x) = x^n, \qquad x \in (-1, 1).$$

   Show that f_n converges, but not uniformly, on $(-1, 1)$. [4]

4. On the interval $[-\pi, \pi]$, the function $f(x) = x^2 - \pi^2$ can be expressed in a Fourier cosine series:

$$f(x) = -\frac{2}{3}\pi^2 + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^2} \cos(kx) \qquad x \in [-\pi, \pi].$$

(a) Does the following equality hold for all $y \in [0, \pi]$ [3]

$$\int_0^y f(x)dx = -\frac{2}{3}\pi^2 y + \sum_{k=1}^{\infty} \frac{4(-1)^k}{k^3} \sin(ky)?$$

(b) Calculate the outcome of the following sum [2]

$$\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)^3}.$$

5. Let (X, ρ) be a metric space.

(a) Give the definition of a Cauchy sequence. [1]

(b) Let $\{x_n, n \in \mathbb{N}\}$ be a convergent sequence in X. Prove that it is a Cauchy sequence. [2]

(c) Let $a \in X$, and $r > 0$. Prove that the set $V_r(a) := \{x \in X \mid r < \rho(x, a) < 2r\}$ is an open subset of X. [3]

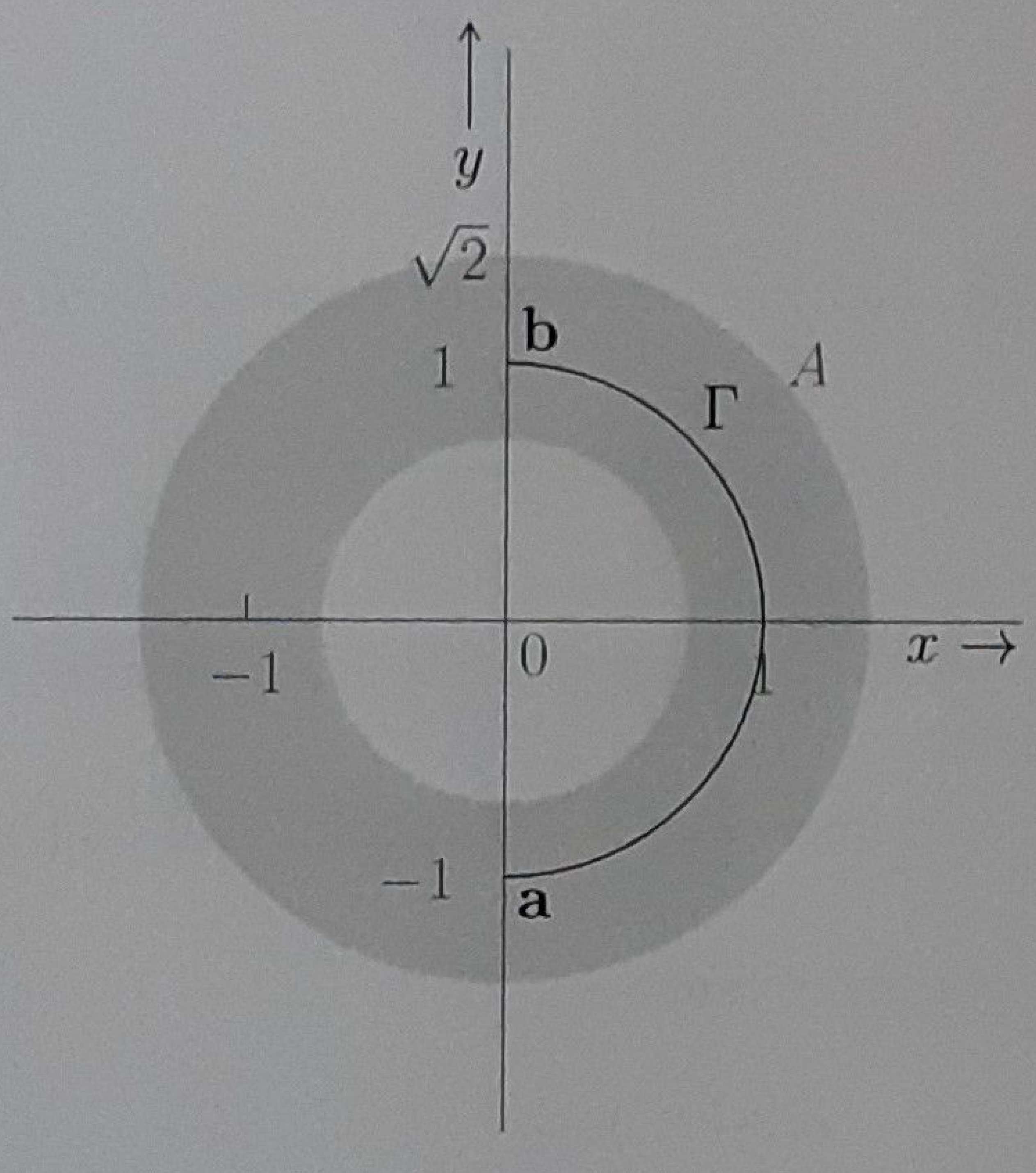

Figure 1: The half-circle Γ within the open set A

6. Consider the function $q : \mathbb{R}^2 \to \mathbb{R}$, given by

$$q(x,y) = \begin{cases} \frac{\sqrt{x^4+y^2}}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) At which points of $\mathbb{R}^2$ is q differentiable? [3]

(b) Is there a point $\mathbf{c}$ on the y-axis such that [2]

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a})?$$

Here $\mathbf{b} = (0,1)$ and $\mathbf{a} = (0,-1)$, see also Figure 1.

7. *Mean Value Theorem on a Circle.* In $\mathbb{R}^2$ we consider the annulus $A = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{2} < x^2 + y^2 < 2\}$. We take the points $\mathbf{b} = (0,1)$, and $\mathbf{a} = (0,-1)$. Finally, let Γ be the half-circle $\Gamma = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } x \geq 0\}$. This is all drawn in Figure 1.

We parametrise Γ as $g(t) = (\sin(t), \cos(t))$ where $t \in [0, \pi]$. Let $f : A \mapsto \mathbb{R}$ be differentiable on A. Prove that there exists a point $\mathbf{c} = (c_1, c_2)$ on Γ such that [3]

$$f(\mathbf{b}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot \begin{bmatrix} c_2 \\ -c_1 \end{bmatrix}.$$

8. Given the function $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$:

$$f(x,y) = \begin{bmatrix} xe^x \cos(y) - ye^x \sin(y) \\ ye^x \cos(y) + xe^x \sin(y) \end{bmatrix}.$$

(a) Prove that there exists an f^{-1} which maps $(e,0)$ to $(1,0)$ and is differentiable in some nonempty open set containing $(e,0)$. Compute the derivative of this function in the point $(e,0)$. [3]

(b) Does the inverse function exist globally? [2]

Grade:	$\frac{\text{score on test}}{38} \times 9 + 1$ (rounded off to one decimal place)