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# Uitwerking Test 1

## Analyse II, 28-9-'21

1a| A series converge if the sequence of partial sums $s_n = \sum_{k=1}^{n} a_k$ converges. (2)

bi| We see that it is an alternating sequence with $\boxed{a_k \geq 0}$. ($\frac{1}{2}$)

Furthermore, $a_k \downarrow 0$, decreasing since

$$\left(\frac{2x - \log(x)}{x^2 + \log(x)}\right)' = \frac{(2 - \frac{1}{x})(x^2 + \log(x))}{} - \frac{(2x + \frac{1}{x})(2x - \log(x))}{(x^2 + \log(x))^2} < 0 \text{ for } x > 1$$

($\frac{1}{2}$)

($\frac{1}{2}$) So $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.

It is not abs. convergent, since by limit comparison test

$$\lim_{k\to\infty} \frac{a_k}{\frac{1}{k}} = 2 \quad \text{and} \quad \sum \frac{1}{k} \text{ is divergent.}$$

($\frac{1}{2}$)



bii| • Since $k \geq 1$, we see $\sin(\frac{1}{k^2})\cos(\frac{1}{k}) \geq 0$. ($\frac{1}{2}$)

• $$\lim_{k\to\infty} \frac{\sin(\frac{1}{k^2})\cos(\frac{1}{k})}{\frac{1}{k^2}} = 1.$$ ($\frac{1}{2}$)

So by limit Comparison Test $\sum_{k=1}^{\infty} a_k < \infty$ ($\frac{1}{2}$)

($\frac{1}{2}$) • Since all terms are positive the series is also absolutely convergent.

Remark You could use Comparison Test since

$$0 \leq \sin(\tfrac{1}{k^2})\cos(\tfrac{1}{k}) \leq \frac{1}{k^2} \cdot 1 = \frac{1}{k^2}.$$

c| Consider even and odd indeces. (1)

$b_\ell := a_{2\ell} \quad c_\ell = a_{2\ell-1}, \ \ell = 1, \ldots$

Then $$\left|\frac{b_{\ell+1}}{b_\ell}\right| = \left|\frac{a_{2\ell+2}}{a_{2\ell}}\right| < \varepsilon < 1$$

So $\sum_{\ell=1}^{\infty} |b_\ell| < \infty$. (1)

Similar

$$\left|\frac{c_{\ell+1}}{c_\ell}\right| = \left|\frac{a_{2\ell+1}}{a_{2\ell-1}}\right| < \varepsilon < 1$$

and thus

$$\sum_{\ell=1}^{\infty} |c_\ell| < \infty$$ ($\frac{1}{2}$)

(1/2) This implies that

$$\sum_{\ell=1}^{\infty} |b_\ell|+|c_\ell| = \sum_{\ell=1}^{\infty}|b_\ell| + \sum_{\ell=1}^{\infty}|c_\ell| < \infty$$

Now $\sum_{k=1}^{\infty} |a_k|$ ~~is a rearrangement of~~ $\cancel{\sum_{\ell=1}^{\infty}}$ equals $\sum_{\ell=1}^{\infty} |c_\ell| + |b_\ell|$ and so converges. ☒

2a) Let $f_k(x) = e^{-k^2 x}$

Then $|f_k(x)| \le e^{-k^2 x_0}$ for $x \ge x_0$ (1)

(1) Now $\sum_{k=1}^{\infty} e^{-k^2 x_0} < \infty$

since $e^{-k^2 x_0} \le e^{-k x_0} = (e^{-x_0})^k$

and $e^{-x_0} < 1$ since $x_0 > 0$.

[Remark: Can be done with integral criterium $0 \le e^{-t^2 x_0}$ and $\int_1^\infty e^{-t^2 x_0}\,dt < \infty$]

(1) So by Weierstrass M-test

$\sum_{k=1}^{\infty} f_k(x)$ converges absolutely and uniformly.

b) Let $x_1 \in (0,\infty)$. Then there exist $x_0 > 0$ s.t. $x_0 < x_1$

$\sum_{k=1}^{\infty} f_k(x)$ conv. unif. on $(x_0,\infty)$



and so pointwise in $(x_0,\infty)$ and thus in $x_1$. (1)

- ~~Take $k \ge N$~~

(2) If it converges uniformly on $(0,\infty)$, there $\forall \varepsilon > 0\ \exists N > 0$ s.t.

$$\left|\sum_{k=n}^{m} f_k(x)\right| < \varepsilon \quad \text{(Cauchy criterion on } s_n(x))$$

for all $x \in (0,\infty)$ and $n,m \ge N$

Choose $\varepsilon > 0$ and assume $N$ exists

Take $x_N = \frac{1}{N^2}$. Take $\varepsilon < e^{-1}$

Take $n = N = m$

$$\sum_{k=n}^{m} f_k(x_N) = f_N(x_N) = e^{-1} \not< \varepsilon.$$

So contradiction.

Remark: You could take

$$\sup_{x>0}\left|\sum_{k=n}^{m} f_k(x)\right| = \sup_{x>0} \sum_{k=n}^{m} f_k(x)$$

$f_k > 0$

$$\le \sum_{k=n}^{m} 1. \quad \times$$

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3a) $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with

$a_k = \frac{k!}{(2k)!} > 0$ $\left(\frac{1}{2}\right)$

$\left(\frac{1}{2}\right)$ $R = \lim_{k\to\infty} \frac{\frac{k!}{(2k)!}}{\frac{(k+1)!}{(2k+2)!}} \overset{\left(\frac{1}{2}\right)}{=} \lim_{k\to\infty} \frac{(2k+2)!\,k!}{(2k)!\,(k+1)!}$

$= \lim_{k\to\infty} \frac{(2k+2)(2k+1)}{k+1} = \infty.$ $(1)$

$\left(\frac{1}{2}\right)$ Hence convergence interval is $\mathbb{R}$.

Remark The root test is hard. Needs Stirlings formula (not in book?).

b) $\left(\cancel{\tfrac{1}{2}}\right)$ Since $f(x)$ is given by a power series on $\mathbb{R}$ centered at $x=0$. It has a converging power series around every point in $\mathbb{R}$ and thus it is analytic. (Theorem 7.46) $\left(\frac{1}{2}\right)$

c) Since $f$ is analytic on $\mathbb{R}$, we know that $a_k = \frac{f^{(k)}(0)}{k!}$

$(1)$ If $f$ would be zero on $(-a,a)$, then $f^{(k)}(0) = 0$

This is contradiction since $\cancel{f}\, a_k \neq 0$.

Remark Easy solution

$f(0) = \frac{0!}{(2 0)!} = 1 \neq 0.$

Remark Easy solution-2

$a_k > 0 \Rightarrow a_k x^k > 0$ when $x > 0$

So $f(x) > 0$ for $x > 0$.