# (Class) Test-2: Analysis II
# Statistics and Analysis (202001350)

26-October-2021, 08:45–10:15, NH-115 Total Points : 20

**All answers must be motivated.**
**Approach to a solution is equally important as the final answer.**
**Use of an electronic calculator or a book is not allowed.**
**Good Luck!**

1. Consider on $\mathbb{Z}$ the following metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } |x-y| = 1 \\ 2 & \text{if } |x-y| \geq 2. \end{cases}$$

   (a) Prove that $d$ defines a metric on $\mathbb{Z}$. [3]

   (b) Let $V \subseteq \mathbb{Z}$. Show that $V$ is an open subset of $\mathbb{Z}$ with respect to the metric $d$. [2]

   (c) Show that any subset $V \subseteq \mathbb{Z}$ is closed with respect to the metric $d$. [1]

   (d) With the metric $d$. Are there subsets of $\mathbb{Z}$ which are closed and bounded, but not compact? [2]

2. Let $V$ be an open and bounded subset of $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $C^1$-function. Prove that there exists a constant $M \geq 0$ such that for all $\mathbf{a}$ and $\mathbf{b}$ in $V$ there holds [4]

$$|f(\mathbf{b}) - f(\mathbf{a})| \leq M\|\mathbf{b} - \mathbf{a}\|, \qquad \mathbf{a}, \mathbf{b} \in V.$$

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x,y) = \begin{cases} \dfrac{\cos(x^2+y^2) - 1}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

   (a) Show that $f$ differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$. [2]

   (b) Determine whether $f$ is differentiable at $(0,0)$. [3]

4. For the following function prove that $f^{-1}$ exists and is differentiable on an open set $W \subset \mathbb{R}^2$ containing the point $(0,0)$. Furthermore compute $D(f^{-1})(0,0)$. [3]

$$f(u,v) = \left(u^2 + e^v - 1, e^u - e^{uv}\right).$$

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| **Grade:** | $\dfrac{\text{score on test}}{20}$ | $\times 9 + 1$ | (rounded off to one decimal place) |