Make-up Exam: Analysis II
Statistics and Analysis (202001350)

November, 10, 2022, 08:45 – 11:45 Total Points : 36

All answers must be motivated.
Approach to a solution is equally important as the final answer.
The exam is closed-book and also, use of an electronic calculator is not allowed
Good Luck!

1. Let the sequence $\{a_k,\ k \in \mathbb{N}\}$ be such that:

$$\text{The infinite series } \sum_{k=1}^{\infty} a_k \text{ converges.} \tag{1}$$

   (a) If $a_k \geq 0,\ k \in \mathbb{N}$ and $\{b_k,\ k \in \mathbb{N}\}$ is a bounded sequence, then prove that [2]

$$\sum_{k=1}^{\infty} a_k b_k < \infty.$$

   (b) Does the statement in part (a) also holds when we do not assume that $a_k \geq 0$? [2]

2. Find the radius of convergence and the convergence interval of the power series: [4]

$$\sum_{k=1}^{\infty} \frac{k \log(k+1)}{\sqrt{2+k^2}} (x-\pi)^k.$$

3. (a) Give the definition of pointwise convergence of a series of real-valued functions, using ϵ-N arguments/language. [1]

   Define the following sequence of functions $f_n : \mathbb{R} \mapsto \mathbb{R}$

$$f_n(x) = \frac{1}{n} \sin\left(\frac{x}{n+1}\right), \qquad x \in \mathbb{R}, \quad n \in \mathbb{N}.$$

   (b) Show that $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise on $\mathbb{R}$. [2]

   (c) Show that $f(x) := \sum_{n=1}^{\infty} f_n(x)$ converges uniformly on every bounded interval of $\mathbb{R}$. [2]

   (d) Determine the value of the integral $\int_{-\pi}^{\pi} f(x)dx$. [2]

   (e) Prove that there exists a constant $M > 0$ such that [2]

$$|f'(x)| \leq M$$

   for all $x \in \mathbb{R}$.

4. Consider on $\mathbb{R}$ the following (candidate) metric

$$d(x,y) = |x^3 - y^3|$$

(a) Prove that for all $x, y \in \mathbb{R}$ there holds [1]

$$d(x,y) \leq d(x,0) + d(0,y)$$

(b) Does d define a metric on $\mathbb{R}$? [2]

5. Let (X, ρ) be a metric space and let E be a subset of X.

(a) Show that when E is bounded, then there exists an $M > 0$ such that for all $v, w \in E$ there holds $\rho(v,w) \leq M$. [3]

(b) Give the definition of the boundary ∂E of the set E. [1]

(c) Prove that the boundary ∂E is a closed subset of X. [2]

6. Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$, given by

$$g(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^4+y^2} & (x,y) \neq (0,0), \\ 1 & (x,y) = (0,0). \end{cases}$$

(a) At which points of $\mathbb{R}^2 \setminus \{0,0\}$ is g continuous? [2]

(b) Is g is continuous at $(0,0)$? [2]

7. Let $\mathcal{O}$ be an open subset of $\mathbb{R}^n$ and let W be a compact and convex subset of $\mathcal{O}$. Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a C^1 function on $\mathcal{O}$. Prove that there exists a $M > 0$ such that [3]

$$|f(\mathbf{b}) - f(\mathbf{a})| \leq M\|\mathbf{b} - \mathbf{a}\| \quad \text{for all } \mathbf{a}, \mathbf{b} \in W.$$

8. Prove that there exist functions $u(z)$, and $v(z)$, and an $r > 0$ such that u, v are continuous differentiable and satisfy the equations

$$\begin{aligned} u^2 + v^2 + z^2 &= 9, \\ 3u + 2v - z &= 1 \end{aligned}$$

on $B_r(1)$ and satisfy $u(1) = 2$, $v(1) = -2$. [3]

Grade:	$\dfrac{\text{score on test}}{36} \times 9 + 1$	(rounded off to one decimal place)