(Class) Test-2: Analysis II
Statistics and Analysis (202001350)

25-October-2022, 08:45 – 10:15, NH-207 Total Points : 20

All answers must be motivated.
Approach to a solution is equally important as the final answer.
Use of an electronic calculator or a book is not allowed.
Good Luck!

1. We define X to be the space consisting of the subsets of $\{1, \ldots, 2022\}$. On X we define the following candidate metric for $V_1, V_2 \subseteq \{1, \ldots, 2022\}$

$$\rho(V_1, V_2) = \text{ number of elements in } V_1 \cap V_2.$$

Does ρ define a metric on X? [2]

2. Let (Y, τ) be a metric space.

   (a) Prove by means of the definition that if $V \subseteq Y$ is compact, then it is bounded. [2]

   (b) Let $\{x_n\}$ be a sequence in Y, and let $y \in Y$ be given.
   Give the definition of $\{x_n\}$ converges (in Y) to y. [2]

   (c) Assume that $\{x_n\}$ converges to y, and let $\mathbf{a}$ be an element of Y.
   Prove that $\tau(x_n, \mathbf{a}) \to \tau(y, \mathbf{a})$ as $n \to \infty$. [3]

3. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a C^1-function.

   Show that $g(t) := \|f(t)\|^2$ (the Euclidian squared norm of $f(t)$) is differentiable on $\mathbb{R}$, and determine its derivative. [3]

4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$f(x, y) = \begin{cases} \dfrac{(x^2 + y^2)\log(x^2 + 1)}{\sin(x^2 + y^2)} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

   (a) Show that f differentiable on $\mathbb{R}^2 \setminus \{(0, 0)\}$. [2]

   (b) Determine whether f is differentiable at $(0, 0)$. [3]

5. Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}$, and consider the function $f : \Omega \to \Omega$ given by

$$f(x_1, x_2) = \left(x_1^4 + x_1^2 x_2^2,\ x_2^4 + x_1^2 x_2^2\right).$$

   Prove that f^{-1} exists and is differentiable on an open set $W \subset \Omega$ containing the point $(2, 2)$. Furthermore, compute $D(f^{-1})(2, 2)$. [3]

Grade:	$\dfrac{\text{score on test}}{20} \times 9 + 1$ (rounded off to one decimal place)