Sample Exam: Analysis-2 (202200237), MOD-02-AM: Structures and Systems

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Date/Time: 18-April-2023, 08:45 – 11:45

- Closed book/calculator exam! May use one single-sided handwritten A4-paper.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A *must* use the four steps (practiced during Tutor Sessions).
  - (i.) Get Started: describe what the problem is [and your initial thoughts]
  - (ii.) Devise Plan: provide an outline of how you plan to solve (or have solved) the problem
  - (iii.) Execute: execute your plan (and try) to reach your solution
  - (iv.) Evaluate: reflect on your solution and/or approach [with something new not yet mentioned]

  Points are distributed (roughly) as: steps (i.)+(ii.) 35%, step (iii.) 50% and step (iv.) 15%.
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.6 \times \text{Grade\_Section\_A} + 0.4 \times \text{Grade\_Section\_C}$ (see Assessment Policy for details)
- Good Luck!

Section C:

Total Points : 30

1. Consider the sequence of functions $f_n : (0, \infty) \to \mathbb{R}$ given by

$$f_n(x) = \frac{x}{n} \sum_{i=0}^{n-1} \sin^2\left(\frac{xi}{n}\right), \quad x > 0, \quad n \in \mathbb{N}.$$

Find the pointwise limit of the sequence, i.e., $\lim_{n\to\infty} f_n(x)$. [6]

[Hint: Think of an appropriate integral.]

2. (a) Let $f : [0, \infty) \to \mathbb{R}$ be such that it has a continuous derivative and

$$\int_0^\infty (f(t) + tf'(t))\, dt = 1.$$

Find the limit: $\lim_{x\to\infty} xf(x)$. [6]

(b) Suppose that $f : [0, \infty) \to \mathbb{R}$ is continuous. Define, for $x > 0$,

$$g(x) := \int_1^{e^x} \frac{(x - \ln t)}{t} f(\ln t)\, dt.$$

i) Show that (for $x > 0$) $\quad g(x) = \int_0^x \int_0^t f(u)\, du\, dt.$ [5]

[Hint: Use a suitable change of variable in the single integral and exchange the order of integration in the double integral.]

ii) Calculate $g''(x)$. [2]

3. (a) Find the interval of convergence I for the power series $f(x) := \sum_{k=0}^{\infty} \frac{1}{\sqrt{k}} x^{2k}$. [5]

(b) Argue that f is analytic on any open interval within I. [1]

(c) Comment on the existence of $f^{(2)}(0.5)$. [1]

4. Let $f, g : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by

$$g(x,y) = \ln(x^2+y^2) + x^4y^3, \quad (x,y) \neq (0,0), \qquad \text{and} \qquad f = \frac{\partial^5 g}{\partial x^4 \partial y}.$$

Show that the function $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ is constant on $\mathbb{R}^2 \setminus \{(0,0)\}$. [4]

Hint: It will help to consider $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$.
You may also use the fact that if a two-variable polynomial $Q(x,y)$ is non-zero on an open domain $D \subseteq \mathbb{R}^2$, then the partial derivatives (of all orders) of $\frac{1}{Q(\cdot,\cdot)}$ exist and are continuous on D.

Section A: [Follow the four-step procedure] Total Points : 20

5. (a) Determine whether the series $\sum_{k=1}^{\infty} \left(\frac{3-(-1)^k}{\pi}\right)^{3k}$ converges. [1+1+1]

(b) Suppose that for all $n \in \mathbb{N}$, $a_n \geq 0$ and $b_n \neq 0$. Prove that [3+3+1]

$$\text{if } \lim_{n\to\infty} \frac{a_n}{b_n} = -1 \text{ and } \sum_{k=1}^{\infty} a_k \text{ converges, then } \sum_{k=1}^{\infty} b_k \text{ converges.}$$

6. The output, $f(x)$, from a system with any input x ($\in \mathbb{R}$) can be characterized by

$$f(x) := \sum_{n=1}^{\infty} \frac{\sin(nx)+\sqrt{n}}{n^2+x^2}, \qquad x \in \mathbb{R}.$$

Prove that f, as a function $f : \mathbb{R} \to \mathbb{R}$, is continuous on $\mathbb{R}$. [4+5+1]