Final Exam: Analysis-2 (202200237), MOD-02-AM: Structures and Systems

Instructors: Carlos Pérez Arancibia and Felix Schwenninger

Date/Time: 29-January-2024, 13:45 – 16:45

- Closed book exam! May use one single-sided handwritten A4-paper.
- All answers must be motivated, including the answers of Section C.
- Answers for Section A could use the four steps (practiced during Tutor Sessions).
- Section Grade: $\dfrac{\text{Obtained score}}{\text{Total points}} \times 9 + 1$ (rounded off to one decimal place)
- Course Grade: $0.4 \times \text{Grade\_Section\_A} + 0.6 \times \text{Grade\_Section\_C}$ (see Assessment Policy for details)
- Good Luck!

Section C:

Total Points : 30

1. Evaluate the limit [10]

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right|$$

where

$$f(t) := \sin(\pi \ln(1+t)).$$

[Hint: Begin by proving the following identity which holds for all functions $f : [0,1] \to \mathbb{R}$ that are continuously differentiable:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left| f\left(\frac{k}{n}\right) - f\left(\frac{k-1}{n}\right) \right| = \int_0^1 |f'(t)|\, dt.$$

Additionally, you could find the inequality $\sqrt{e} < 2$ helpful.]

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x,y) := \begin{cases} x^2 y \sin\left(\dfrac{1}{x^2+y^2}\right) & \text{when } (x,y) \neq (0,0), \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

   (a) Prove from the definition that f is continuous at $(0,0)$. [3]

   (b) Compute the partial derivatives $\dfrac{\partial f}{\partial x}(x,y)$ and $\dfrac{\partial f}{\partial y}(x,y)$. [3]

   (c) Is f differentiable at $(0,0)$? [4]

3. Consider the integral:

$$I(\epsilon) := \iint_{D(\epsilon)} \frac{1}{x+y} dA, \quad 0 < \epsilon < 1,$$

where $D(\epsilon) \subset \mathbb{R}^2$ is the region bounded by the lines: $x + y = \epsilon$, $x + y = 1$, $x = 0$, and $y = 0$.

(a) Evaluate $I(\epsilon)$ directly using Fubini's theorem. Does the improper integral $\lim_{\epsilon\to 0+} I(\epsilon)$ exist? [3]

(b) Evaluate $I(\epsilon)$ using the substitution $(x, y) = (u - uv, uv)$. [3]

(c) Evaluate $I(\epsilon)$ using the substitution $(x, y) = (u\cos v, u\sin v)$. [4]

[Hint: The trigonometric identity $\sin v + \cos v = \sqrt{2}\sin\left(v + \frac{\pi}{4}\right)$ could be helpful.]

Section A:

Total Points : 20

4. (a) Find a closed-form expression for the series [5]

$$\sum_{k=1}^{\infty} \frac{2k}{k+1}(1-x)^k$$

and determine the largest set on which such expression is valid.

(b) Let $\alpha \geqslant 0$ and define $u_n : [0, \infty) \to \mathbb{R}$ by [5]

$$u_n(x) = \frac{x^\alpha}{1 + n^2x^2}, \quad n \in \mathbb{N}.$$

Show that $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on $[0, 1]$ if $\alpha > 1$.

5. (a) Prove that if $f(t)$ is a continuous function on $[0, 1]$, then $\int_0^1 |f(t)|dt = 0$ implies that $f(t) = 0$ for all $t \in [0, 1]$. [5]

(b) Let $f(t)$ be a differentiable function on $[0, 1]$, and $f'(t)$ is integrable on $[0, 1]$. Assume that $f(0) = 0$ and $|f'(t)| \leq |f(t)|$ for all $t \in (0, 1)$. Prove that $f(t) = 0$ for all $t \in (0, 1)$. [5]

[Hint: Utilize the Fundamental Theorem of Calculus and consider changing the order of integration over a certain triangular region.]