Test T1 Differential Equations & Numerical Methods

Module	: AM M6 Dynamical Systems (201500103)
Date	: Tuesday December 4, 2018
Time	: 8:45 - 11:45 uur
Duration	: 180 min (In case of extra time: 225 min)
	: 30 min (In case only Numerical Methods is tested)
	: 150 min (In case only Differential Equations is tested)
Module-coordinator	: H.G.E. Meijer
Examinator	: H.G.E. Meijer
Test Type	: Closed book
Supplements	: None
Tools allowed	: (Grafical) Calculator

Remarks:

- Motivate your answers.
- This test consists of 3 pages, including this one, and contains 7 exercises.
- For this test you can get 36 points; i.e. grade = 1+points/4. The points for each exercises are mentioned below.
- If you only take Differential Equations, please skip Exc 6 & 7;
  If you only take Numerical Methods, hand in Exc 6 & 7 only. The grading is adjusted accordingly.
- Only use UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Subpoints:

1	4	4b	1	5b	2	6a	1
2	3	4c	2	5c	3	6b	1
3	6	4d	2	5d	3	7a	3
4a	1	5a	1	5e	2	7b	1

Grade = 1+points/4

Exercises Differential Equations

Exercise 1. Solve the following initial value problem

$$\frac{dx}{dt} = x(1-2t) \qquad x(0) = -1. \tag{1}$$

Also state the maximal interval of existence.

Exercise 2. Sketch the phase line for

$$y' = (y-2)(y^2-1)\cos(y). \tag{2}$$

Restrict your plot to the interval $|y| < 3$.

Exercise 3. We define the following matrix and vectors:

$$A = \begin{pmatrix} 7 & -4 \\ 8 & -5 \end{pmatrix}, \quad b(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Solve $\frac{dx}{dt} = Ax(t) + b(t)$ for $x(0) = x_0$.

Exercise 4. Consider the following system

$$\begin{cases} x' = & -y, \\ y' = & x - x^3 - y^3. \end{cases} \tag{3}$$

(a) Give the definition of Lyapunov stability.

(b) Show that $V = x^2 + y^2 - \frac{1}{2}x^4$ is a Lyapunov function.

(c) Show that the origin is asymptotically stable for (3).

(d) Not all orbits in the plane converge to the origin. Give an argument why that is the case.

Exercise 5. We consider the following system

$$\begin{cases} \dot{x} & = & (x+y)(1-y^2), \\ \dot{y} & = & (y-x)(1-x^2). \end{cases} \tag{4}$$

(a) Show the vector field f is invariant by a rotation over $\pi/2$. That is, $f(S(x,y)) = S(f(x,y))$ for $S := (x,y) \to (-y,x)$.

(b) Sketch all nullclines. Indicate the direction of the vector field at each segment.

(c) Determine all equilibria and determine their type. You may use symmetry to minimize the amount of calculations. Sketch the phase portrait for two qualitatively different equilibria.

We now define a region S using the following polygon. The inner boundary is the square given by the lines $x = \pm 1$ and $y = \pm 1$. For the outer boundary we start with straight lines from $(-3,-3)$ to $(-1,-3)$ to $(2, -3\frac{1}{2})$ to $(3,-3)$. So we have three lines; $y = -3$, $y = -3 - (x+1)/6$ and $y = -\frac{7}{2} + \frac{x-2}{2}$ with corresponding values of x. Next we extend it using the symmetry to get a closed outer boundary.

T.O.P. for 5de and Numerical Methods

(d) Show that S is positively invariant.
Hint: Your estimates will involve two cubic polynomials. You may use that these cubic polynomials have two complex roots, and one real. Determine the real root with at least one digit accuracy for your estimate. Due to symmetry it suffices to do this in this sector only.

(e) Can you prove the existence of a periodic orbit for system (4)? If yes, do it. If not, what would you need to prove the existence of this periodic orbit?

Exercises Numerical Methods

Exercise 6.

(a) What is the expression for the condition number of the problem: 'compute the value of the function f in a point x'? Compute the condition number in case f is given by

$$f(x) = \sqrt{1-x} \quad ; \quad x \leq 1$$

(b) Consider $x = 1 + \varepsilon$ with $0 \leq \varepsilon \ll 1$. Assume x is specified in $N > 1$ significant digits. For what values of ε does the answer $y = f(x)$ have no significant digits?

Exercise 7. The numerical approximation of the first derivative of a smooth function f on a stencil of five grid points can be expressed as

$$D_1(h) = \frac{-f(x_0+2h) + 8f(x_0+h) - 8f(x_0-h) + f(x_0-2h)}{12h} = f'(x_0) + C\,h^4$$

where $h > 0$ and $f'(x_0)$ denotes the exact first derivative of f in x_0.

(a) Determine the expression for C. What is the order of accuracy of this method?

(b) We approximate the first derivative numerically on a computer. Is the error dominated by round-off errors or by truncation errors? Distinguish in your answer the case $h \approx \epsilon_C$ and the case $h \gg \epsilon_C$.