Test T1 Differential Equations & Numerical Methods

Module	: AM M6 Dynamical Systems (201500103)
Date	: Friday December 21, 2018
Time	: 8:45 - 11:45 uur
Duration	: 180 min (In case of extra time: 225 min)
	: 30 min (In case only Numerical Methods is tested)
	: 150 min (In case only Differential Equations is tested)
Module-coordinator	: H.G.E. Meijer
Examinator	: H.G.E. Meijer
Test Type	: Closed book
Supplements	: None
Tools allowed	: (Graphical) Calculator

Remarks:

- Motivate your answers.
- This test consists of 3 pages, including this one, and contains 5 exercises.
- For this test you can get 36 points; i.e. grade = 1+points/4. The points for each exercise are mentioned below.
- If you only take Differential Equations, please skip Exc 5;
  If you only take Numerical Methods, hand in Exc 5 only. The grading is adjusted accordingly.
- Use UT exam paper only. Write your name and student number on each sheet of paper. Do not hand in your notes or scratch paper.

Subpoints:

1a	2	3a	2	4b	2	5a	1.5
1b	4	3b	1	4c	2	5b	1.5
2a	6	3c	2	4d	3	5c	2
2b	1	4a	2	4e	3	5d	1

Grade = 1+points/4

Exercises Differential Equations

Exercise 1. Consider the following differential equation

$$\frac{dx(t)}{dt} = \frac{1}{\sin(x(t))}. \tag{1}$$

(a) Sketch the direction field for (1) for $-4 < x < 4$, and include a few solutions.

(b) Solve (1) with the initial condition $x(0) = \frac{\pi}{2}$. Also state the maximal interval for t, as well as the range of $x(t)$.

Exercise 2. Define the matrix

$$A = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

(a) Compute e^{tA}.

(b) Characterize all $x \in \mathbb{R}^3$ such that $\lim_{t\to\infty} e^{tA}x = 0$.

Exercise 3. We consider the following system with parameter b

$$\begin{cases} \dot{x} &= 2x + y - bx(x^2+y^2) + x(x^2+y^2)^2, \\ \dot{y} &= 2y - x - by(x^2+y^2) + y(x^2+y^2)^2. \end{cases} \tag{3}$$

(a) Transform the system to polar coordinates.

(b) Determine the value(s) of b such that system (3) has exactly one periodic orbit.

(c) Sketch the phase plane in the (x, y)-plane for $b = 1$ and $b = 3$.

Exercise 4. Consider the following system with parameter a

$$\begin{cases} x' = & -y, \\ y' = & x + \frac{1}{2}ay^2 - x^3. \end{cases} \tag{4}$$

(a) Determine the type of the three equilibria, for all $a \in \mathbb{R}$.

(b) For $a = 0$, show that system (4) is Hamiltonian, and also determine a Hamiltonian.

(c) Plot the phase portrait for $a = 0$.

(d) Show that $E(x, y) = e^{ax}\left(a^4y^2 - 2a^3x^3 + 6a^2x^2 + 2a^3x - 12ax + 12 - 2a^2\right)$ is a conserved quantity for system (4) for all $a \neq 0$.

(e) Plot the phase portrait for $a = 1$ and $a = -1$.
Hint: The positions of the extrema of $E(x, y)$ along the x-axis, i.e., the equilibria, do not change.

T.O.P. for Numerical Methods

Exercises Numerical Methods

Exercise 5. Using the trapezoidal rule for integration of the function

$$f(x) = 2e^{-x-x^2} + g(x), \quad \text{where} \quad g(x) = (3e^{-2} - 1)x^2$$

on the interval $[0, 1]$, i.e.,

$$I = \int_0^1 f(x)\mathrm{d}x$$

we obtain numerical approximations $I(h)$ at step size h as given in the following table:

h	numerical value $I(h)$
0.5000	0.817286388000510
0.2500	0.816212596855330
0.1250	0.816148425997833
0.0625	0.816144458517675

(a) Determine from these data the order of convergence of this proces, i.e., determine the value of p in the relation

$$I(h) = I + a_p h^p + O(h^{p+1}).$$

(b) Determine the best approximation for I by extrapolating once. Also specify an estimate for the absolute error.

(c) Explain the value of p as determined in (a) using the following theorem.

Theorem: For a sufficiently differentiable function f the approximation $I(h)$ for the integral as obtained by the trapezoidal rule obeys

$$I(h) = \int_0^1 f(x)dx + a_2h^2 + a_4h^4 + \ldots + a_{2m}h^{2m} + O(h^{2m+2}),$$

where

$$a_{2k} = \frac{b_{2k}}{(2k)!}(f^{(2k-1)}(1) - f^{(2k-1)}(0))$$

and known numbers b_{2k}.

(d) Using the same theorem, what would you expect for the error after one extrapolation, if the trapezoidal rule would be applied to the function g only?