Test T1 Differential Equations & Numerical Methods

Module	: AM M6 Dynamical Systems (201500103)
Date	: Tuesday December 3, 2019
Time	: 8:45 - 11:45 uur
Duration	: 180 min (In case of extra time: 225 min)
	: 30 min (In case only Numerical Methods is tested)
	: 150 min (In case only Differential Equations is tested)
Module-coordinator	: B.G. Geurts
Lecturer DE	: H.G.E. Meijer
Test Type	: Closed book
Supplements	: None
Tools allowed	: (Graphical) Calculator

Remarks:

- Motivate your answers.
- This test consists of 3 pages, including this one, and contains 6 exercises.
- For this test you can get 36 points, i.e., the grade = 1+points/4. The points for each exercises are mentioned below.
- If you only take Differential Equations, please skip Exc 5 & 6;
  If you only take Numerical Methods, hand in Exc 5 & 6 only. The grading is adjusted accordingly.
- Only use UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Subpoints:

1	4	3a	1	4a	2	5a	1.5
2a	2	3b	2	4b	2	5b	1.5
2b	6	3c	2	4c	3	6a	1.5
2c	1	3d	2	4d	3	6b	1.5

Grade = 1+points/4

Exercises Differential Equations

Exercise 1. Solve the following initial value problem

$$\frac{dx}{dt} = \sqrt{x-1}, \qquad x(0) = 2. \tag{1}$$

Also state the maximal interval of existence.

Exercise 2. Consider the matrix and vector

$$A = \begin{pmatrix} 3 & 1 & -4 \\ 1 & -3 & -2 \\ 2 & 1 & -3 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}, \quad a \in \mathbb{R}.$$

(a) Show using the characteristic polynomial that the eigenvalues of A are $\{1, -2, -2\}$.

(b) Determine e^{At}.

(c) Determine a such that $\lim_{t\to\infty} e^{At} x_0 = 0$.

Exercise 3. Consider the following system

$$\begin{cases} x' = & x(2 - x - 2y), \\ y' = & y(2 - 2x - y). \end{cases} \tag{2}$$

(a) Determine the equilibria.

(b) Determine the type of the equilibria.

(c) Show that if an orbit starts on the line $x = y$, then it stays on that line, i.e., the line is invariant.

(d) Sketch the complete phase portrait of (2) for $0 \leq x, y \leq 3$. Include the nullclines in your sketch.

Exercise 4. We consider the following system

$$\begin{cases} \dot{x} &= -y + x(2 - x^2 - y^2) + bx^2 y, \\ \dot{y} &= x + y(2 - x^2 - y^2) - bx^3. \end{cases} \tag{3}$$

(a) Derive the transformation from Euclidean to polar coordinates in the following way. Start with the relation $r(t)e^{i\theta(t)} = x(t) + iy(t)$ and differentiate this relation w.r.t. time. Next multiply both sides by $(x - iy)$ and determine $\dot{r}$ and $\dot{\theta}$ from this relation.

(b) Convert system (3) to polar coordinates.

(c) Use a theorem to prove that (3) has a periodic orbit if $b < \frac{1}{2}$.

(d) Sketch phase portraits for $b = 0$ and $b = 1$ in the (x, y)-plane.

T.O.P. for Numerical Methods

Exercises Numerical Methods

Exercise 5. The numerical approximation of the second derivative of a smooth function f can be expressed using Taylor expansion as

$$D_2(h) = \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2} \quad = \; f^{(2)}(x_0) + C\,h^2 + O(h^4),$$

where $h > 0$ and $f^{(2)}(x_0)$ denotes the exact second derivative of f in x_0.

(a) Determine the expression for C.

(b) There exists a value h_c for h at which the total error is minimal for a given function f. What is the dominant error contribution in case $h \ll h_c$? And what is it in case $h \gg h_c$?

Exercise 6. A numerical approximation of a quantity $I(h)$ with a small parameter h yields a sequence of values given in the following table:

h	numerical value $I(h)$
$1/2$	3.26914555200204
$1/4$	3.26485038742132
$1/8$	3.26459370399133
$1/16$	3.26457783407070

(a) Determine from these data the order of convergence of this proces, i.e., determine the value of p in the relation

$$I(h) = I + a_p h^p + O(h^{p+1}).$$

(b) Determine the best approximation for I by extrapolating once. Also specify an estimate for the absolute error.