Test T1 Differential Equations & Numerical Methods

Module	: AM M6 Dynamical Systems (201500103)
Date	: Friday December 20, 2019
Time	: 8:45 - 11:45 uur
Duration	: 180 min (In case of 25% extra time: 225 min)
	: 30 min (In case only Numerical Methods is tested)
	: 150 min (In case only Differential Equations is tested)
Module-coordinator	: B.G. Geurts
Lecturer DE	: H.G.E. Meijer
Test Type	: Closed book
Supplements	: None
Tools allowed	: (Graphical) Calculator

Remarks:

- Motivate your answers.
- This test consists of 3 pages, including this one, and contains 7 exercises.
- For this test you can get 36 points, i.e., the grade = 1+points/4. The points for each exercises are mentioned below.
- If you only take Differential Equations, please skip Exc 6 & 7; Grade=1+9*points/30
  If you only take Numerical Methods, hand in Exc 6 & 7 only. The grading is adjusted accordingly.
- Only use UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Subpoints:

1	4	3a	2	4b	2	6a	1.5
2a	5	3b	2	4c	3	6b	1.5
2b	2	3c	3	5	4	7a	1.5
2c	1	4a	2			7b	1.5

Grade = 1+points/4

Exercises Differential Equations

Exercise 1. Solve the following initial value problem

$$\frac{dx}{dt} + \cos(t)x(t) = \frac{1}{2}\sin(2t), \qquad x(0) = 1. \tag{1}$$

Also state the maximal interval of existence, and plot the solution for $0 \le t \le 10$.

Exercise 2. Consider the matrix and vector

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 0 \end{pmatrix}, \quad x_0 = \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

(a) Determine e^{At}.

(b) Plot the phase portrait of $x' = Ax$.

(c) Determine a such that $\lim_{t\to\infty} e^{At}x_0 = 0$.

Exercise 3. Consider the following system with parameter $b \in \mathbb{R}$

$$\begin{cases} x' = & bx + y - 3x(x^2+y^2) + x(x^2+y^2)^2, \\ y' = & by - x - 3y(x^2+y^2) + y(x^2+y^2)^2. \end{cases} \tag{2}$$

(a) Convert system (2) to polar coordinates.
If you can't solve this, then proceed with $r' = r(b + r + 2r^2)$ and $\theta' = 1$.

(b) Determine the range of b such that (2) has two periodic orbits.

(c) Sketch the phase portrait of (2) for $b = -1$, $b = 1$ and $b = 3$.

Exercise 4. We consider the following system

$$\begin{cases} \dot{x} &= y(y-x), \\ \dot{y} &= x(x-y) - y^3. \end{cases} \tag{3}$$

(a) Formulate the theorem for Lyapunov stability, including the properties for a Lyapunov function.

(b) Show that the origin is stable using the function $V = x^2 + y^2$.

(c) Prove that the origin is asymptotically stable for (3).

Exercise 5. Prove that the following planar system

$$\begin{cases} \dot{x} &= -y(2+xy), \\ \dot{y} &= 2x + y - yx^2 - y^3. \end{cases} \tag{4}$$

has a periodic orbit.

T.O.P. for Numerical Methods

Exercises Numerical Methods

1. **Exercise 6.**

   (a) What is the expression for the condition number of the problem: 'compute the value of the function f in a point x'? Compute the condition number in case f is given by

   $$f(x) = \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

   (b) Consider $x = 0.68$ with a possible absolute error of 0.01. We wish to compute $f(x)$.

   (b1) What is the value of the condition number in this case?

   (b2) Given the condition number from (b1), what can you conclude regarding the relative error with which $f(0.68)$ is computed?

2. **Exercise 7.** With the help of a numerical integration process we obtain for a certain integral I the following approximations $I(h)$ as a function of the stepsize h:

	h	$I(h)$
A	1/2	0.79438324164397
B	1/4	0.85164722970527
C	1/8	0.86270616289176
D	1/16	0.86522491674297
E	1/32	0.86582874610414

   (a) Determine the order p of the numerical integration process based on the data in the table, i.e., determine the value of p from the expression $I(h) = I + c\,h^p + O(h^{p+1})$, $p \in \mathbb{N}$.

   (b) Perform one extrapolation to obtain an improved approximation for I. Include an estimate of the absolute error.