Test Differential Equations

Module	: AM M6 Dynamical Systems (202001353)
Course	: Differential Equations (202001354)
Date	: Friday December 4, 2020
Time	: 9:45 - 12:15 uur
Duration	: 150 min (In case of extra time: 187 min)
Module-coordinator	: B.J. Geurts
Teacher	: H.G.E. Meijer

Test Type	: Closed book
Supplements	: None
Tools allowed	: (Graphical) Calculator

Notices:

- Motivate your answers with proper argumentation and calculations
- This test consists of 2 pages, including this one, and contains 5 exercises.
- Only use UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.
- For this test you can get 36 points, i.e., your grade equals 1+points/4.
  The points for each exercise are mentioned below.

Subpoints:

1a	3	1d	2	3b	2	4c	2
1b	2	2	6	4a	4	5	7
1c	2	3a	4	4b	2		

Exercises Differential Equations

Exercise 1. Consider the following differential equation

$$\frac{dx}{dt} = \begin{cases} \sqrt{1 - x(t)^2}, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases} \tag{1}$$

a. Determine a solution of (1) satisfying the initial condition $x(0) = 0$.
   Hint: If you do not recognize the integral, consider a suitable substitution of variables.

b. Sketch this solution as a function of time, and provide the maximal domain of existence of this non-constant solution.

c. Explain your result using the theorem on solutions for initial value problems.

d. Formulate an extension of this non-constant solution with $x(0) = 0$ that exists for all $t \in \mathbb{R}$.

Exercise 2. Compute e^{tA} for the matrix A defined by

$$A := \begin{pmatrix} -4 & 0 & 0 \\ 1 & -3 & -1 \\ 1 & 0 & -3 \end{pmatrix}.$$

Exercise 3. Consider the differential equation $x' = (x^2 - 4)(x^2 - 1) + a$, with parameter $a \in \mathbb{R}$.

a. For $a = 0$, determine the equilibria and classify their type, also sketch the phase line.

b. Determine all values for a such that the system has two equilibria.

Exercise 4. A closed habitat with preys and predators is modelled with the following system

$$\begin{cases} x' = & x\left(2 - \dfrac{y}{1+x} - x\right), \\ y' = & y\left(-\dfrac{1}{2} + \dfrac{x}{1+x}\right), \end{cases}$$

with $0 \leq x, y \leq 4$.

a. Determine all equilibria and classify them.

b. Sketch the nullclines in the first quadrant, and add orbits near the equilibria based on the linearization.

c. Show that the triangle with $x, y \geq 0$ and $x + y \leq 4$ is positive invariant.

Exercise 5. Consider the following planar system

$$\begin{cases} x' = & y + x - x^3 - 3xy^2, \\ y' = & y - x - y^3. \end{cases} \tag{2}$$

Assume the origin is the only equilibrium of this system. Prove that the system (2) has a periodic orbit.
Hint: Consider circles with a suitable radius.