Exercises Numerical Mathematics

Exercise 1.

Consider the two-point boundary value problem with variable coefficients:

$$-\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right) + \gamma(x)u(x) = f(x) \quad ; \quad 0 < x < 1 \tag{1}$$

with boundary conditions $u(0) = d_0$; $u(1) = d_1$. Here d_0 and d_1 are assigned constants and α, γ and f are smooth functions such that $\gamma(x) \geq 0$ and $\alpha(x) \geq \alpha_0 > 0$.

We discretize this problem on a uniform grid $x_k, k = 0, \ldots, n$ consisting of $n+1$ points, separated by a grid spacing $h = 1/n$. The corresponding $n+1$ values $\{u_k\}$ are approximations of $u(x_k)$.

(a) Introduce a second grid consisting of the midpoints $x_{k+1/2} = (x_k + x_{k+1})/2$ of the intervals $[x_k, x_{k+1}]$. Use Taylor expansion around $x_{k+1/2}$ to show that

$$\alpha_{k+1/2}\frac{u_{k+1} - u_k}{h} = \alpha(x_{k+1/2})\frac{du}{dx}(x_{k+1/2}) + \beta h^2 u'''(\xi_k)$$

for suitable $\xi_k \in [x_k, x_{k+1}]$. Derive an expression for β.

It may be shown that the finite difference operator

$$\left(D_2 u\right)_k = \frac{1}{h}\left(\alpha_{k+1/2}\frac{u_{k+1} - u_k}{h} - \alpha_{k-1/2}\frac{u_k - u_{k-1}}{h}\right)$$

approximates $\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right)$ in the location x_k with second order accuracy.

(b) Introduce the vector of unknowns $\mathbf{u} = [u_0, \ldots, u_n]$ and the correspondingly $\mathbf{f} = [f_0, \ldots, f_n]$. The discretization based on D_2 can be written as $A\mathbf{u} = \mathbf{f}$. Specify in detail the corresponding matrix A, including the treatment of the boundary conditions and the discrete equations.

Exercise 2.

With the help of a numerical integration process we obtain for a certain integral I the following approximations $I(h)$ as function of step size h:

h	$I(h)$
1/2	3.26914555200204
1/4	3.26485038742132
1/8	3.26459370399133
1/16	3.26457783407070

(a) Determine the order p of the numerical integration process based on the data in the table, i.e., determine the value of p from the expression $I(h) = I + ch^p + O(h^{p+1})$, $p \in N$

(b) Perform one extrapolation to obtain an improved approximation for I. Include an estimate of the absolute error and present your final answer in its significant digits only.