AM-M6-NUM: Numerical Mathematics

Course	: AM-M6 - Numerical Mathematics (202001356)
Module	: Dynamical Systems
Date	: Thursday, February 3, 2022
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Time	: 08:45 - 10:45 (11:15)
Duration	: 120 min (in case of extra time: 150 min)

Notice:

- Always motivate your answers.
- This test consists of 3 pages, including this one, and contains 3 exercises.
- For this test you can get a grade = 1+#points with maximally 9 points distributed over the exercise as detailed below.
- Use only UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Points rewarded:

Exercise	Points
1a	1.5
1b	1.5
2a	2
2b	1
3a	1
3b	1
3c	1

Grade = 1+#points

Exam Questions Numerical Mathematics

Exercise 1.

Consider the two-point boundary value problem with variable coefficient:

$$-\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right) + u(x) = \cos(2\pi x) \quad ; \quad 0 < x < 1 \tag{1}$$

with boundary conditions

$$\frac{du(0)}{dx} = 1 \; ; \; u(1) = 0$$

Here, α and f are smooth functions with $\alpha(x) \geq \alpha_0 > 0$.

We discretize this problem on a uniform grid $x_k, k = 0, \ldots, n$ consisting of $n+1$ points, separated by a grid spacing $h = 1/n$. The corresponding $n+1$ values $\{u_k\}$ are approximations of $u(x_k)$.

(a) Introduce a second grid consisting of the midpoints $x_{k+1/2} = (x_k + x_{k+1})/2$ of the intervals $[x_k, x_{k+1}]$. Use Taylor expansion around $x_{k+1/2}$ to show that

$$\alpha_{k+1/2}\frac{u_{k+1} - u_k}{h} = \alpha(x_{k+1/2})\frac{du}{dx}(x_{k+1/2}) + \beta h^2 \frac{d^3u}{dx^3}(x_{k+1/2}) + O(h^3)$$

Derive an expression for β.

It may be shown that the finite difference operator

$$\left(D_2 u\right)_k = \frac{1}{h}\left(\alpha_{k+1/2}\frac{u_{k+1} - u_k}{h} - \alpha_{k-1/2}\frac{u_k - u_{k-1}}{h}\right)$$

approximates $\frac{d}{dx}\left(\alpha(x)\frac{du(x)}{dx}\right)$ in the location x_k with second order accuracy.

(b) Introduce the vector of unknowns $\mathbf{u} = [u_0, \ldots, u_n]$ and the correspondingly $\mathbf{f} = [f_0, \ldots, f_n]$. The discretization based on D_2 can be written as $A\mathbf{u} = \mathbf{f}$. Specify in detail the corresponding matrix A, including the treatment of the boundary conditions and the discrete equations.

Exercise 2.

With the help of a numerical integration process we obtain for a certain integral I the following approximations $I(h)$ as function of step size h:

h	$I(h)$
1/2	3.26914555200204
1/4	3.26485038742132
1/8	3.26459370399133
1/16	3.26457783407070

(a) Determine the order p of the numerical integration process based on the data in the table, i.e., determine the value of p from the expression $I(h) = I + c\,h^p + O(h^{p+1})$, $p \in N$

(b) Perform one extrapolation to obtain an improved approximation for I. Include an estimate of the absolute error and present your final answer in its significant digits only.

Exercise 3.

Consider the smooth function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \exp(-x^2) - x^2$$

a) Sketch f and df/dx on the interval $[-1, 1]$ and prove that f has exactly two roots.

We compute the positive root of f by iteration.

b) Starting from the interval $[0, 1]$, how many steps will the bisection method require to approximate the root with an accuracy of 10^{-4}.

c) Formulate Newton's method to determine the positive root of f and compute the first iteration starting from $x_0 = 1$.