**Exercises Differential Equations**

**Exercise 1.** Consider the following initial value problem

$$2t\frac{dx}{dt} = -x(t) + t, \qquad x(4) = \frac{3}{2}. \tag{1}$$

(a) Provide the general solution for the homogeneous part.

(b) Solve the initial value problem.

(c) State the (maximal) interval of existence of the solution.

**Exercise 2.** Consider the matrix

$$A = \begin{pmatrix} -1 & 2 & 2 \\ -1 & 2 & 3 \\ 1 & -1 & -2 \end{pmatrix}.$$

(a) Show that $-1$ and $+1$ are the eigenvalues of $A$. Indicate the multiplicity of each.

(b) Determine $e^{At}$.

(c) Specify all vectors $b$ such that $\lim_{t\to\infty} e^{At}b = 0$.

**Exercise 3.** We consider a population model of coexisting species

$$\begin{cases} x' = & x(5 - 2x - y), \\ y' = & y(4 - x - 2y). \end{cases} \tag{2}$$

(a) Determine the four equilibria and classify their type.

(b) Sketch the nullclines in the first quadrant, and add the direction of the vector field on the nullclines. Sketch characteristic orbits near the equilibria.

**Exercise 4.** Consider the following system

$$\begin{cases} x' = & -y - x^3, \\ y' = & 2x^3 - y^3. \end{cases} \tag{3}$$

(a) Show that linearization does not allow to determine the stability of the origin for (3).

(b) Formulate the theorem for Lyapunov stability, including the properties for a Lyapunovtion. function

(c) Investigate the stability of the origin for (3) using $V = x^4 + y^2$.

**Exercise 5.** We study the following population model with harvest

$$\begin{cases} x' & = & x(3-x) - \dfrac{hx}{1+3x}, \end{cases} \tag{4}$$

where $h \geq 0$ is the harvesting rate of the population $x > 0$. The harvest goes to zero if the population is small, and is finite if the population is large.

(a) Determine the equilibria $x^*$ as a function of $h$.

(b) Give an interval for $h$ such that we have exactly two positive equilibria.

(c) For $h = 8$, determine the stability of the equilibrium at $x = 1$.

(d) Sketch the phase line of (4) for $h = 2$, $h = 6$ and $h = 10$.