AM-M6-NUM: Numerical Mathematics

Course	: AM-M6 - Numerical Mathematics (202001356)
Module	: Dynamical Systems
Date	: Thursday, January 17, 2023
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Time	: 13:45 - 15:45 (15:15)
Duration	: 120 min (in case of extra time: 150 min)

Notice:

- Always motivate your answers.
- This test consists of 2 pages, including this one, and contains 3 exercises.
- For this test you can get a grade = 1+#points with maximally 9 points distributed over the exercise as detailed below.
- Use only UT exam paper. Write your name and student number on each sheet of paper. Do not hand in your notes on scratch paper.

Points rewarded:

Exercise	Points
1a	1
1b.i	0.5
1b.ii	1
1c.i	1
1c.ii	0.5
2a	0.5
2b	1
2c	0.5
2d	1
3a	1
3b	1

Grade = 1+#points

Exam Questions Numerical Mathematics

Exercise 1.

Consider the function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \cos(x) + e^{-x} \tag{1}$$

a) Prove that f has at least one root on the interval $[0, \pi]$.

We approximate one of the possible roots of f numerically using iteration.

b) i) What numerical method would you select to guarantee that you will surely find one root on the interval $[0, \pi]$?
   ii) How many iteration steps are needed with this method to approximate a root with a guaranteed accuracy of 10^{-4}?

c) An efficient method to determine roots is Newton's method.
   i) Formulate Newton's method to approximate a root of f.
   ii) Starting from $x = 0$ what approximation to a root of f do you obtain after one step for the function f in (1)?

Exercise 2.

We wish to approximate $y(1)$ where the function y satisfies

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad f(x, y) = -2y \quad ; \quad y(0) = 3 \tag{2}$$

The two-stage Runge-Kutta method is a well-known method to solve (2) numerically. Denoting with y_n the numerical approximation of $y(x_n)$ in $x_n = nh$, where h is the step-size, then this method proceeds as follows

$$\tilde{y}_{n+1/2} = y_n + \frac{h}{2} f(x_n, y_n), \quad y_{n+1} = y_n + hf(x_n + \frac{h}{2}, \tilde{y}_{n+1/2}).$$

(a) Is this method implicit or explicit?

(b) Compute y_1 and y_2 in detail.

(c) The local truncation error of this method is of third order. What is the order of the global truncation error? 2

(d) How would you design a numerical experiment with which you may determine the global truncation error of this Runge-Kutta method? Please specify the separate steps needed to verify the theoretical prediction.

Exercise 3.

We want to approximate a quantity $I(0)$ and obtain a sequence of numerical estimates $I(h)$ at step sizes h as given in the following table:

h	$I(h)$
1/2	3.26914555200204
1/4	3.26485038742132
1/8	3.26459370399133
1/16	3.26457783407070

(a) Determine from these values the order of convergence of this process, i.e., determine the value of p in the relation

$$I(h) = I(0) + ah^p + O(h^{p+1}).$$

(b) Determine the best approximation for $I(0)$ from this information by extrapolating once. Also, specify an estimate for the absolute error.