Numerical Mathematics (202200241), 2022-2023
Exam, June 23rd 2023, 8.45-11.45

All of your answers need to be justified. The use of electronic devices is not allowed.

Exercise 1. Let $m, n \in \mathbb{N}$. Denote $\|x\|_\infty = \max_{1\le j\le n} |x_j|$ the max-norm for $x \in \mathbb{R}^n$. Let $\boldsymbol{A} = (a_{i,j}) \in \mathbb{R}^{m\times n}$, $1 \le i \le m$, $1 \le j \le n$, be a matrix. Show that the induced matrix norm is given by

$$\|\boldsymbol{A}\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{i,j}|.$$

Exercise 2. We search the linear model $g(t) = a + bt$ that fits the data points

t_i	1	2	3	4
g_i	2	6	8	8

(i) Formulate the corresponding least-squares problem to determine $a, b \in \mathbb{R}$.
(ii) Determine the least-squares solution a, b.
(iii) Show that a, b depend continuously on g_i in the max-norm.

Exercise 3. Let $a \in \mathbb{R}$ with $a > 0$. We search the positive solution $x \in \mathbb{R}$ of the equation

$$x^2 = a. \tag{1}$$

(i) Show that Newton's method applied to (1) gives the sequence $\{x_k\}_{k\in\mathbb{N}}$ defined by

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right), \qquad x_0 > \sqrt{a},$$

provided that all $x_k \neq 0\ \forall k \in \mathbb{N}_0$.
(ii) Use mathematical induction to show that $x_k > \sqrt{a}$ for all $k \in \mathbb{N}_0$.
(iii) Show that $\{x_k\}_{k\ge0}$ is strictly decreasing. [redacted]
(iv) Show that x_k converges to $\sqrt{a}$ (**Hint:** Use (ii) and (iii).)
(v) Show that $\{x_k\}_{k\in\mathbb{N}}$ converges quadratically to $\sqrt{a}$.

Exercise 4. (i) Compute the LU-decomposition of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -1 & 3 \end{pmatrix}.$$

(ii) Use the LU-decomposition calculated in (i) to compute the determinant of $\boldsymbol{A}$.
(iii) Use the LU-decomposition calculated in (i) to compute the inverse of $\boldsymbol{A}$.
(iv) Compute the condition number of $\boldsymbol{A}$ with respect to the max-norm $\|\cdot\|_\infty$. (**Hint:** Ex. 1.)

Exercise 5. For the approximation of $\int_0^1 f(x)dx$ consider the quadrature formula

$$Q(f) = w_0 f(0) + w_1 f(x_1).$$

(i) Determine w_0, w_1 and x_1 such that the degree of exactness is at least two.
(ii) Determine the degree of exactness of $Q(f)$.

Exercise 6. Let $t_i = ih$ for $h > 0$ and $i \in \mathbb{N}_0$, and let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous and Lipschitz-continuous in the second argument. Consider the implicit Euler method

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1}), \quad i \ge 0, \tag{2}$$

to approximate the solution $y(t)$ to $y'(t) = f(t, y(t))$, $t > 0$, and $y(0) = y_0 \in \mathbb{R}$ given.
(i) Give a condition on h such that y_{i+1} in (2) exists and is unique.
Hint: For which $h > 0$ is $\Psi(y) = y_i + hf(t_{i+1}, y)$ a contraction?
(ii) Determine the order of consistency of (2).

Exercise	1.	2.	3.	4.	5.	6.	total	Grade
Points	4	1+2+1	2+3+1+2+2	2+1+4+1	3+1	3+3	36	(Points + 4)/4