Numerical Mathematics (202200241), 2022-2023
Resit, July 7th 2023, 8.45-11.45

All of your answers need to be justified. The use of electronic devices is not allowed.

Exercise 1. (i) Let $m, n \in \mathbb{N}$ and let $\boldsymbol{A} = (a_{i,j}) \in \mathbb{R}^{m\times n}$, $1 \le i \le m$, $1 \le j \le n$, be a matrix. Denote by

$$\|\boldsymbol{A}\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{i,j}|^2}, \tag{1}$$

the Frobenius norm for $\boldsymbol{A} \in \mathbb{R}^{m\times n}$. Show that

$$\|\boldsymbol{A}\|_F = \sqrt{\operatorname{trace}\left(\boldsymbol{A}^T\boldsymbol{A}\right)},$$

where trace($\boldsymbol{A}$) is defined to be the sum of all diagonal elements of $\boldsymbol{A}$.

(ii) Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$. Show that for any matrix norm $\|\cdot\|_M$, which is consistent to vector norm $\|\cdot\|_V$, it holds that

$$\rho(\boldsymbol{A}) \le \|\boldsymbol{A}\|_M,$$

where $\rho(\boldsymbol{A}) := \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \boldsymbol{A}\}$ is the spectral radius of $\boldsymbol{A}$.

Exercise 2. We search the model $g(t) = a + b\sin(t) + c\cos(t)$ that fits the data points

t_i	0	$\frac{\pi}{4}$	π	$\frac{5}{4}\pi$
g_i	7	2	1	2

(i) Formulate the corresponding least-squares problem to determine $a, b, c \in \mathbb{R}$.
(ii) Determine the least-squares solution a, b, c.
(iii) Show that a, b, c depend continuously on g_i in the max-norm.

Exercise 3. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = \frac{1}{2}\exp(x/2)$ be given. Show that there is exactly one solution of $x = f(x)$ in the interval $[0, 1]$ by performing the following steps:
(i) Show that $0 \le f(x) \le 1$ for all $x \in [0, 1]$.
(ii) Show that f is a contraction on $[0, 1]$.
(iii) Formulate the Newton iteration for solving $x - f(x) = 0$, and perform the first step for $x_0 = 0$.

Exercise 4. (i) Compute the LU-decomposition of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}.$$

(ii) Use the LU-decomposition calculated in (i) to compute the determinant of $\boldsymbol{A}$.
(iii) Use the LU-decomposition calculated in (i) to compute the inverse of $\boldsymbol{A}$.
(iv) Compute the condition number of $\boldsymbol{A}$ with respect to the Frobenius norm. (**Hint:** Eq. (1).)

Exercise 5. For the approximation of $\int_0^1 f(x)dx$ consider the quadrature formula

$$Q(f) = w_0 f\left(\frac{1}{2} - \frac{1}{2\sqrt{3}}\right) + w_1 f\left(\frac{1}{2} + \frac{1}{2\sqrt{3}}\right).$$

(i) Determine w_0, $w_1 \in \mathbb{R}$ such that all polynomials of degree less or equal to one are integrated exactly.
(ii) Determine the degree of exactness of $Q(f)$.

Exercise 6. Let $t_i = ih$ for $h > 0$ and $i \in \mathbb{N}_0$, and let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be continuous. Consider the modified Euler method

$$y_{i+1/2} = y_i + \frac{h}{2} f(t_i, y_i), \qquad i \ge 0, \tag{2a}$$

$$y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_{i+1}), \quad i \ge 0, \tag{2b}$$

to approximate the solution $y(t)$ to $y'(t) = f(t, y(t))$, $t > 0$, and $y(0) = y_0 \in \mathbb{R}$ given.
(i) Show that (2) is a one-step method and give the increment function.
(ii) Show that (2) is consistent.
(iii) Determine the order of consistency of (2) assuming sufficient regularity of f.

Exercise	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**	**total**	**Grade**
Points	2+2	2+2+1	2+2+2	2+1+4+1	3+4	1+1+4	36	(Points + 4)/4