Systems Theory (202001355) – Exam

Date:	15-01-2024
Time:	8:45–11:45 (till 12:30 for students with special rights)
Place:	SP5
Course coordinator:	Gjerrit Meinsma
Allowed aids during test:	a basic calculator

1. Consider the system described by the differential equation

$$\ddot{y}(t) - 2\dot{y}(t) - 3y(t) = \beta\dot{u}(t) + u(t). \tag{1}$$

   (a) Determine an equivalent state model

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{2}$$

   (Equivalent means: (u, y) satisfies (1) if-and-only-if there is an x such that (u, x, y) satisfies (2).)

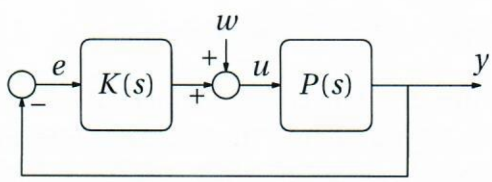
   (b) For which β is state model (2) controllable?

   (c) For which β is state model (2) stabilizable?

   (d) Let $\beta = 1$. Determine an F such that $A - BF$ has eigenvalues -1 (twice).

   (e) Take $\beta = 1$. Determine a controller with input y and output u that makes the closed loop system asymptotically stable.

2. Consider the system with disturbance w:



   (a) Determine the transfer function from w to u.

   (b) Suppose $P(s) = 1, K(s) = 1/s$, and that $w(t) = \sin(t)$. Determine numbers A, ω, b such that $\lim_{t\to\infty} y(t) - A\sin(\omega t + b) = 0$.

   (c) Suppose $P(s) = 1$ and $w(t) = \sin(t)$, and that $K(s)$ stabilizes the closed loop system and such that $\lim_{t\to\infty} y(t) = 0$. Show that $K(s)$ must have the form

$$K(s) = \frac{N_K(s)}{(s^2+1)D_K(s)}$$

   for certain polynomials $N_K(s), D_K(s)$.

   (d) Suppose $P(s) = 1$, and that $w(t) = \sin(t)$. Determine a stabilizing controller $K(s)$ such that $\lim_{t\to\infty} y(t) = 0$.

3. Consider

$$\dot{x} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -2 \end{bmatrix} x$$

   (a) Determine the Kalman Observability Decomposition of this system.

   (b) Determine the impulse response of this system (assumed initially-at-rest).

4. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{k\times n}$. Consider the **discrete time** system

$$x(t+1) = Ax(t), \qquad t \in \mathbb{Z},$$
$$y(t) = Cx(t).$$

   This discrete-time system is said to be *observable* if there is a $T \in \mathbb{N}$ such that $x(0)$ follows uniquely from $y(0), y(1), \dots, y(T)$.

   Prove that the discrete-time system is observable if-and-only-if the *observability matrix* (as we know from our course) has rank n.

5. Three questions.

   (a) What is the *frequency response* of the system described by $\dot{x} = Ax + Bu, y = Cx + Du$?

   (b) Is the system $y(t) = \mathrm{e}^t u(t)$ linear?

   (c) Is the system $y(t) = \mathrm{e}^t u(t)$ time-invariant?

6. *LQ control.* Let $b, c \in \mathbb{R}$ and both nonzero. Consider the scalar system

$$\dot{x}(t) = bu(t), \quad y(t) = cx(t), \quad x(0) = x_0$$

   and standard LQ cost $\int_0^\infty y^2(t) + u^2(t)\,\mathrm{d}t$.

   (a) Determine the input $u(t)$ (as a function of $x(t), b, c$) that minimizes the LQ cost over all stabilizing inputs, and determine the eigenvalue of the corresponding closed loop. *Hint: the Algebraic Riccati Equation is* $A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C - PBB^{\mathrm{T}}P = 0$.

   (b) What happens with the optimal cost as $c \to \infty$, and explain in words why this makes sense.

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Exam grade: $1 + 9p/36$.