Course Num. PDEs. Practice test Jan. 8, 2015

Solutions

A1 A numerical scheme is called convergent if for any fixed point (x^*, t^*) in the computational domain we have:

From

(1) $\Delta t \to 0$, $\Delta x \to 0$ and $\mu = \frac{\Delta t}{(\Delta x)^2} = \text{const}$

(2) $x_j \to x^*$ and $t_n \to t^*$

it follows that

$$U_j^n \to u(x^*, t^*)$$

1 point

A2 let $e_j^n = U_j^n - u(x_j, t_n)$ be (local) error in the numerical scheme

Then $E^n = \max\{|e_j^n|, \ j = \underbrace{0, 1, 2, \ldots, J}_{\text{all mesh points}}\}$

0.5 point

Let $\overline{T}$ = an upper bound for the truncation error, i.e. $|T_j^n| \leq \overline{T}$. Let us set $\overline{T} = C\Delta t$

0.5 point

An estimate holds:

$$E^n \leq n\overline{T}\Delta t \overset{\boxed{n\Delta t = t_F}}{=}$$

therefore:

$$= \overline{T}\, t_F = C\Delta t\, t_F$$

1 point

(since CFL number is fixed, so is C)

A3 Rewrite the given scheme:

$$U_j^{n+1} - U_j^n = \frac{\Delta t}{(\Delta x)^2}\left[D_{j+1/2}(U_{j+1}^n - U_j^n) - D_{j-1/2}(U_j^n - U_{j-1}^n)\right] =$$

$$= \frac{\Delta t}{(\Delta x)^2}\left[D_{j+1/2}U_{j+1}^n - (D_{j+1/2} + D_{j-1/2})U_j^n + D_{j-1/2}U_{j-1}^n\right]$$

with $\mu = \frac{\Delta t}{(\Delta x)^2}$:

$$U_j^{n+1} = \mu D_{j-1/2}U_{j-1}^n + \overbrace{\left(1 - \mu(D_{j-1/2} + D_{j+1/2})\right)}^{\text{denote by } b_j} U_j^n + \mu D_{j+1/2} U_{j+1}^n$$ 1pt.

$\mu D_{j\pm 1/2}$ can be seen as a local CFL number

$$\begin{cases} \mu D_{j\pm 1/2} > 0 \\ b_j^n > 0 \text{ provided the CFL number is small enough} \end{cases}$$ 1pt

positivity test

Sum. property test:

$$\mu D_{j-1/2} + b_j + \mu D_{j+1/2} = 1$$ 1pt

B1 CFL condition is satisfied if the domain of dependence of the equation lies within the domain of dependence of the numerical scheme. 1pt.

The CFL condition can be written as a bound imposed on the CFL number, for this case

$$\nu = a\frac{\Delta t}{\Delta x}$$

B2 Truncation error is the residual obtained when the exact solution of the PDE is substituted into the numerical scheme. 1pt

Let $u(x,t)$ be exact solution of the PDE

$$u_t + a u_x = 0.$$

Consider u_j^{n+1} and u_j^n and omit the superindices n and $n+1$:

(derivative with respect to the space variable x)

$$u_j = u_{j+1/2} + u'_{j+1/2}\frac{-h}{2} + u''_{j+1/2}\frac{h^2}{4} - u'''_{j+1/2}\frac{h^3}{6} + O(h^4)$$ with $h = \Delta x$

Similarly for u_{j+1}^n and u_{j+1}^{n+1}:

$$u_{j+1} = u_{j+1/2} + u'_{j+1/2}\frac{h}{2} + u''_{j+1/2}\frac{h^2}{4} + u'''_{j+1/2}\frac{h^3}{6} + O(h^4)$$

Hence:

$$u_j + u_{j+1} = 2u_{j+1/2} + 2u''_{j+1/2}\frac{h^2}{4} + O(h^4)$$ for $t = t_n$ and $t = t_{n+1}$

Therefore

$$\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right) = 2u_{j+1/2}^{n+1} + 2\left(u''_{j+1/2}\right)^{n+1}\frac{h^2}{4} + O(h^4) -$$
$$-2u_{j+1/2}^n - 2\left(u''_{j+1/2}\right)^n\frac{h^2}{4} + O(h^4)$$

and

$$\frac{\left(u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_j^n + u_{j+1}^n\right)}{2\Delta t} = \frac{u_{j+1/2}^{n+1} - u_{j+1/2}^n}{\Delta t} + \frac{\left(u''_{j+1/2}\right)^{n+1} - \left(u''_{j+1/2}\right)^n}{\Delta t}\cdot\frac{h^2}{4} + O(h^4)$$

$$= \left(\dot{u} + O(\Delta t)^2\right)_{j+1/2}^{n+1/2} + \left(\dot{u}'' + O(\Delta t)^2\right)_{j+1/2}^{n+1/2}\frac{h^2}{4} + O(h^4)\frac{1}{\Delta t}$$

(derivative with respect to t)

$\boxed{1 \text{ point}}$

Due to the symmetry in the space and time of the two fractions in formula (3), we have

$$\frac{\left(u_{j+1}^n + u_{j+1}^{n+1}\right) - \left(u_j^n + u_j^{n+1}\right)}{2\Delta x} = \left(u' + O(\Delta x)^2\right)_{j+1/2}^{n+1/2} + \underbrace{\left(\ddot{u}' + O(\Delta x)^2\right)_{j+1/2}^{n+1/2}\frac{(\Delta t)^2}{4}}_{O(\Delta t)^2} + \underbrace{\frac{1}{\Delta x}O(\Delta t)^3}_{O(\Delta t)^2}$$

if CFL number is bounded

Adding up the two expressions, we get

$$\frac{\cdots}{2\Delta t} + a\frac{\cdots}{2\Delta x} = \underbrace{\left(\dot{u} + au'\right)_{j+1/2}^{n+1/2}}_{=0} + O(\Delta x)^2 + O(\Delta t)^2 =$$

$=0$ because $u(x,t)$ is the solution

$$= O(\Delta x)^2 + O(\Delta t)^2$$ ← second order in space and time.

$\boxed{1 \text{ point}}$

B3 damping error $= |\lambda| - 1$ ← exact damping = no damping

phase error $= \arg(\lambda) - (-\nu\xi)$

↑ exact shift in phase

Thus to evaluate the errors we need to estimate $|\lambda|$ and $\arg(\lambda)$ — 1 point

$\lambda = 1 - \nu + \nu \cdot e^{-i\xi} = 1 + \nu(\cos\xi - i\sin\xi) - \nu = 1 + \nu\cos\xi - i\nu\sin\xi - \nu$

$|\lambda|^2 = (1 - \nu + \nu\cos\xi)^2 + \nu^2\sin^2\xi =$

$= (1-\nu)^2 + 2\nu(1-\nu)\cos\xi + \nu^2(\sin^2\xi + \cos^2\xi) =$ — 0.5 point

$= 1 - 2\nu + \nu^2 + 2\nu(1-\nu)\cos\xi + \nu^2 =$

$1 - 2\nu + 2\nu^2 + 2\nu(1-\nu)\left(1 - 2\sin^2\frac{\xi}{2}\right) =$

$= 1 - 2\nu(1-\nu) + 2\nu(1-\nu)\left(1 - 2\sin^2\frac{\xi}{2}\right) =$

$= 1 + 2\nu(1-\nu)\left[-1 + 1 - 2\sin^2\frac{\xi}{2}\right] =$

$= 1 + 2\nu(1-\nu)(-2)\sin^2\frac{\xi}{2} = 1 - 4\nu(1-\nu)\sin^2\frac{\xi}{2}$

$= 1 - O(\xi^2)$ — 0.5 point

$\Rightarrow$ $\underbrace{\text{damping error} = O(\xi^2)}$, $p = 2$.

damping error $= |\lambda| - 1 = \sqrt{1 - O(\xi^2)} - 1 = 1 - O(\xi^2) - 1 =$

$= -O(\xi^2) = O(\xi^2)$