MSc. Course 191551150 | **Exam** | 17.01.2019
Numerical Techniques for PDEs | 13:45–15:45

Please write your answers for the A and B parts on separate sheets of paper, indicating your name and student number on both A and B parts. The use of calculators or other electronic devices is not allowed. Motivate all your answers.

Part A. Parabolic PDEs

Consider the advection-diffusion equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x$$

where the diffusion term is characterized by $a > 0$ and the advection term by a 'transporting velocity' $b > 0$. We set ourselves the goal to develop an *explicit* upwind scheme on a uniform grid with mesh size Δx for this equation.

1pt **A1** Explain the notion of 'wind' in relation to the advection term. In case $b > 0$, what direction would you assign to the wind?

2pt **A2** Formulate the upwind discretization for the advection term and determine the order of accuracy of this discretization.

3pt **A3** Formulate a discretization for the total advection-diffusion equation, based on central discretization for the diffusion term and upwind discretization for the advection term. Determine a stability time-step limitation Δt using the maximum principle. Under what condition on a, b and Δx will advection be the dominant limitation on the stability time-step? Give an interpretation of this condition.

Part B. Hyperbolic PDEs

Consider the scalar conservation law with periodic boundary conditions

$$\begin{aligned} u_t + au_x &= 0 \quad \text{for } x \in (0, 2\pi), t > 0 \\ u(0,t) &= u(2\pi, t), \quad t > 0 \\ u(x,0) &= u_0(x) \quad \text{for } x \in (0, 2\pi), \end{aligned}$$

where $a > 0$ is constant, and u_0 is a 2π-periodic initial datum.
We employ an approximation $U_j^n \approx u(x_j, t_n)$, $j = 1, \ldots, J$.

3pt **B1** Perform a von Neumann (Fourier) stability analysis for the implicit scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = 0, \quad j = 1, \ldots, J. \tag{1}$$

complemented by the periodicity condition $U_{J+1}^n := U_1^n$ and $U_0^n := U_J^n$ for all n. For which CFL numbers $\nu = a\Delta t/\Delta x$ is the scheme (1) stable, respectively, unstable?
Formulate a convergence statement for the scheme (1). What is the order of convergence?

1pt **B2** Show that the scheme (1) conserves mass, i.e., verify that

$$\sum_{j=1}^{J} U_j^n = \sum_{j=1}^{J} U_j^{n+1}.$$

2pt **B3** For the scheme (1), use energy estimates to show the stability estimate

$$\|U^{n+1}\|_2 \leq \|U^n\|_2.$$

Hint: Multiply (1) by U_j^{n+1} and sum over $j = 1, \ldots, J$.
Recall: $\|U^n\|_2 = \sqrt{\sum_{j=1}^{J} |U_j^n|^2}$.

The grade for the test is determined as $G = 1 + 9P/12$ where P is the number of points earned.