MSc. Course
191551150

**Exam**
**Numerical Techniques for PDEs**

16.01.2020
13:45–15:45

Please write your answers for the A and B parts on separate sheets of paper, indicating your name and student number on both A and B parts. The use of calculators or other electronic devices is not allowed. Motivate all your answers.

## Part A. Parabolic PDEs

Consider an advection–diffusion equation

$$u_t = au_{xx} + bu_x + cu, \tag{1}$$

supplied with initial and boundary conditions, where $u = u(x,t)$ is unknown and the functions $a = a(x) > 0$, $b = b(x)$ and $c = c(x,t) \leqslant 0$ are given. We use the following scheme to solve the equation:

$$\frac{U_j^{n+1} - U_j^{n-1}}{2\Delta t} = a_j \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{(\Delta x)^2} + b_j \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{1}{2}(c_j^{n-1}U_j^{n-1} + c_j^{n+1}U_j^{n+1}), \tag{2}$$

where, as usual, the sub- and superindices are related to space and time, respectively.

1pt **A1** With respect to which terms is the given scheme explicit or implicit? Would you expect the scheme to be efficient for advection dominated problems ($a \ll b$) or for diffusion dominated problems ($a \gg b$)? Motivate your answer.

2pt **A2** What is the accuracy order of the scheme (2)? First, answer this question without carrying out a detailed analysis, by recognizing the discretization of the individual terms. Discuss the three cases in which only one of the functions $a$, $b$, $c$ is nonzero. After that, prove the stated accuracy order for the case $c(x,t) < 0$ and $a$ and $b$ are zero functions.

3pt **A3** Show that the scheme (2) can written in the form $\boldsymbol{A}\boldsymbol{U}^{n+1} = \boldsymbol{b}$ where $\boldsymbol{A}$ is a matrix and $\boldsymbol{U}^{n+1}$ is a vector containing the values of the numerical solution at the time level $n+1$, i.e., $U_1^{n+1}, U_2^{n+1}, \ldots$. Specify $\boldsymbol{b}$ and determine the entries of $\boldsymbol{A}$ in a row $j$, where $j$ is a mesh node away from the boundary. How would you solve the linear system $\boldsymbol{A}\boldsymbol{U}^{n+1} = \boldsymbol{b}$?

## Part B. Hyperbolic PDEs

Consider the scalar conservation law with periodic boundary conditions

$$\begin{aligned} u_t(x,t) + au_x(x,t) &= 0 \quad \text{for } x \in (0, 2\pi), t > 0 \\ u(0,t) &= u(2\pi, t), \quad t > 0 \\ u(x,0) &= u_0(x) \quad \text{for } x \in (0,2\pi), \end{aligned} \tag{3}$$

where $a > 0$ is constant, and $u_0$ is a smooth $2\pi$-periodic initial datum.
We employ approximations $U_j^n \approx u(x_j, t_n)$, $j = 0, \ldots, J$ using the standard notation from the lectures, for example $x_j = j\Delta x$ and $t_n = n\Delta t$ with $\mu = \Delta t/\Delta x$ constant. In the following a *second* order scheme should be constructed, which is of the form

$$U_j^{n+1} = c_0 U_j^n + c_{-1}U_{j-1}^n + c_{-2}U_{j-2}^n. \tag{4}$$

2pt **B1** Use Taylor series expansion to construct a second order accurate approximation of $u_x(x_j,t)$ and a first order accurate approximation of $u_{xx}(x_j,t)$ using only $u(x_j,t)$, $u(x_{j-1},t)$ and $u(x_{j-2},t)$, i.e., find $A, B, C, D, E, F \in \mathbb{R}$ such that

$$u_x(x_j,t) = \frac{Au(x_j,t) + Bu(x_{j-1},t) + Cu(x_{j-2},t)}{2\Delta x} + O(\Delta x^2), \tag{5}$$

$$u_{xx}(x_j,t) = \frac{Du(x_j,t) + Eu(x_{j-1},t) + Fu(x_{j-2},t)}{\Delta x^2} + O(\Delta x). \tag{6}$$

**Hint:** $A + B + C = 0$ and $D + E + F = 0$.

2pt **B2** Use Taylor series expansion of $u(x, t+\Delta t)$ at $(x,t)$ to construct a second order accurate scheme of the form (4). In doing so, use (3) to replace temporal derivatives of $u$ by spatial derivatives of $u$, and use your results from **B1**. Give explicitly $c_0$, $c_{-1}$ and $c_{-2}$ in terms of $\nu$.

2pt **B3** Use the CFL condition to derive conditions on the CFL number $\nu = a\Delta t/\Delta x$ that are necessary for convergence of (4).

The grade for the test is determined as $G = 1 + 9P/12$ where $P$ is the number of points earned.