Re-Exam 2, Module 7, Code 201400433

Discrete Structures & Efficient Algorithms

Friday, April 15, 2016, 08:45 - 11:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4) per topic (L&M,ALG,DM). Also if you cannot solve a part of an question you may use that result in subsequent parts of the question.

This exam consists of three parts, with the following (estimated) times:

Languages & Machines (L&M)	1h	(30 points)
Algebra (ALG)	1h 40 min	(50 points)
Discrete Mathematics (DM)	20 min	(10 points)

Total of 30+50+10=90 points. Including 10 bonus points that makes 100 points. Your exam grade is the total number pf points divided by 10.

Please use a new sheet of paper for each part (L&M/ALG/DW)!

Languages & Machines

1. (8 points) Consider the following context-free grammar (CFG) G:

$$G = \begin{cases} S & \to & A\,B \\ A & \to & a \mid a\,A \\ B & \to & C \mid b\,B \\ C & \to & \lambda \mid c\,B \end{cases}$$

   (a) Transform G stepwise to an equivalent CFG G_1, such that G_1 contains neither chain rules, nor λ-rules.

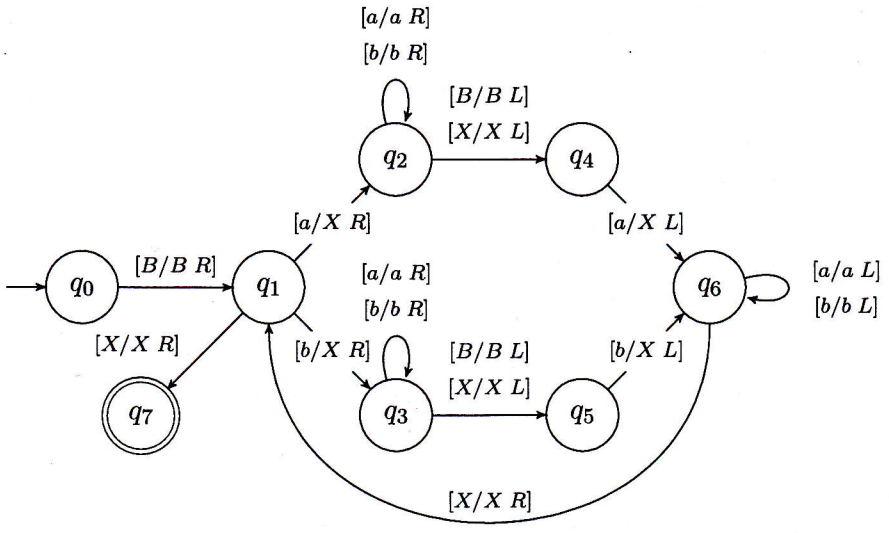
   (b) Provide an equivalent grammar G_2 in Greibach Normal Form.

2. (12 points) Consider the context-free language $L := \{a^i\,b\,a^j \mid j \geq i \geq 0\}$.

   (a) Provide a *deterministic* PDA (stack automaton) for L.
   Explain *shortly* the working of your automaton.

   (b) Is the language $L \cap \big((aaa)^*\,b\,(aaa)^*\big)$ context-free? How does this follow from the closure properties of context-free languages?

3. (10 points) Consider the following Turing Machine (TM).



(a) Given input $abaa$, we write the start configuration as $[q_0BabaaB]$.
What is the end configuration after the TM halts on this input?
Will the word $abaa$ be accepted by this TM?

(b) Which language will be *decided* by this TM? (Explain shortly).

Algebra

4. Let $V = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, be the Klein four-group . As is well-known, each finite group is isomorphic to a subgroup of S_n (the permutationgroup of n symbols).

   (a) (4 points) Why can V not be isomorphic to a subgroup of S_3?

   (b) (6 points) Determine a subgroup H of S_4 such that V is isomorphic to H.

5. Let $(G, \cdot)$ be a group. Define

$$\mathrm{Z}(G) = \{h \in G \mid \forall g \in G : \quad g \cdot h = h \cdot g\}.$$

   (a) (5 points) Show that $Z(G)$ is a subgroup of G.

   (b) (6 points) Now let G be the matrixgroup with as operation matrixmultiplication

$$G = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \quad ad - bc \neq 0\}.$$

   Determine $\mathrm{Z}(G)$.

   (c) (6 points) Show that $\mathrm{Z}(G)$ from part 5b is isomorphic to $\mathbb{R}\backslash\{0\}$ with the usual multiplication.

6. Let $p(x) \in \mathbb{Z}_5[x]$ be given by: $p(x) = x^3 + 2x^2 + 1$ and $I =< p(x) >$ the ideal in $\mathbb{Z}_5[x]$ generated by $p(x)$.

(a) (3 points) Show that $p(x)$ is irreducible.

(b) (4 points) Explain that $\mathbb{F} = \mathbb{Z}_5[x]/I$ is a field.

(c) (4 points) Describe the general form of the elements of $\mathbb{F} = \mathbb{Z}_5[x]/I$. How many different elements are there?

(d) (8 points) Determine the inverse of $2x + 3 + I$ in $\mathbb{F}$.

(e) (4 points) Show that $\mathbb{F}$ is isomorphic to $\mathbb{Z}_5[x]/< x^3 + 3x + 2 >$.

Discrete Mathematics

1. (6 points) Consider the RSA method, and assume that Alice has published the modulus $n = 55$ and the exponent $e = 7$. Bob sends the ciphertext $C = 2$ to Alice. Explain how eavesdropper Eve can compute the original message M, and what she needs for that. Compute M.

2. (4 points) Assume we can do integer division w. rest for any $n, a \in \mathbb{Z}$, $n \geq a$ in time $O(\log n)$. That means we can compute, in $O(\log n)$ time, $q, a \in \mathbb{Z}$ with $n = qa + r$, with $0 \leq r < a$. Denote the function that returns r in $n = qa + r$, `r(n,a)` (in python `r(n,a) = n % a`).

   Describe in pseudocode (no python necessary) an algorithm that determines, for any input $k \in \mathbb{Z}$, if k is a prime or not. Also give an upper bound on the computation time (use $O(\ )$-notation). Is this a polynomial time algorithm?