Kenmerk: EWI2017/TW/DMMP/003/MU_E

Exam 1, Module 7, Code 201600270
Discrete Structures & Efficient Algorithms
Thursday, March 16, 2017, 13:45 - 16:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4) per topic (ADS, DM, L&M).

This exam consists of three parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	1h	(30 points)
Discrete Mathematics (DW)	1h 20 min	(40 points)
Languages & Machines (L&M)	40 min	(20 points)

Total of 30+40+20=90 points. Including 10 bonus points that makes 100 points. Your exam grade is the total number of points divided by 10.

Please use a new sheet of paper for each part (ADS/DW/L&M)!

Algorithms & Data Structures

1. *(10 points)* Consider the following algorithm (with * for multiplication, // for integer division (eg. 7//2 = 3), and **2 for square):

```
def func(n):
    if n==0:
        return 1
    else:
        if n<4:
            return n
        else:
            return 2*func(n//4) + 6 + func(n//4)**2
```

   (a) Give a recursive expression for the time complexity of this algorithm, expressed in the number of arithmetical operations.

   (b) What is the complexity class of this algorithm?

2. *(10 points)*

   (a) Suppose in a heap you update an arbitrary element (say with index i). Describe (in words or in pseudocode) an algorithm that repairs (if necessary) the heap property.

   (b) Given a binary search tree with positive keys, and a key k that does not occur in the tree. Give a function that yields: the biggest key in the tree, smaller than k (or zero if there is no such key). Hint: traverse the tree as if you want to insert k, and keep track of what you encounter.

3. *(10 points)* Suppose you want to put songs on a cd. Suppose you can choose from n songs, where song i takes t_i minutes. You want to fill the cd as much as possible, which means that you want to put as much minutes of music on it as possible. Assume a cd may contain at most 80 minutes of music.

   (a) suppose $C(i, k)$ indicates the minimal remainder (so the amount of unused minutes) if still k minutes need to be filled with songs chosen from the set $\{1, \ldots, i\}$. Explain that

   $$C(i, k) = min\{C(i-1, k), C(i-1, k-t_i)\}$$

   (b) Give a polynomial algorithm, based on dynamic programming, that calculates the maximal amount of minutes you can put on the cd.

Discrete Mathematics

4. (5 points) For given and fixed integer numbers $a, b \in \mathbb{Z}$, assume that we know that there exist $s, t, x, y \in \mathbb{Z}$ so that $as + bt = 6$ en $ax + by = 35$. Prove that a and b are relatively prime.

5. (10 points)

   (a) Let us denote by a_n the number of strings in $\{a, b, c\}^*$ of length n that contain an even number of a's. Compute a_1, a_2, and give a recurrence relation for a_n, $n \geq 3$. (You do not have to solve the recurrence relation.)

   (b) Compute the solution to the recurrence relation

   $$a_n - 6a_{n-1} + 9a_{n-2} = 4n + 4 \quad (n \geq 2) \quad \text{with } a_0 = 5 \text{ and } a_1 = 9\,.$$

6. We want to divide 60 computer science students, 20 mathematics students and 20 atlas students into two tutorial groups of equal size (50), so that in each group we have at least 20 and at most 35 computer science students, at least 5 mathematics students, and at least 5 atlas students.

   (a) (6 points) How many different compositions are there to form a tutorial group (so that both groups fulfil all requirements)? Use a generating function to obtain your result.

   (b) (2 points) How many different ways are there to divide the students among the tutorial groups?[1]

7. (7 points) Let $G = (V, E)$ be a simple, undirected graph, and $\bar{G}$ be the complement graph of G (that has the same set of vertices as G and contains exactly all edges that are not in G). Show that, if both G and $\bar{G}$ are planar, then it must be true that $|V| \leq 10$.

8. (10 points) Suppose we are given a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of (directed) arcs, and $c_a \geq 0$, $a \in A$ are the arc capacities. Also, let $s, t \in V$ and $f = (f_a)_{a \in A}$ be a feasible flow in G. Give a short proof or give a counterexample for each of the following statements.

   (a) f is a maximal (s, t)-flow $\Rightarrow$ $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

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[1] For example, the following two assignments are effectively the same division: [group 1: 35cs+5math+10atlas, group 2: 25cs+15math+10atlas], and [group 1: 25cs+15math+10atlas, group 2: 35cs+5math+10atlas].

(b) There is a maximal (s,t)-flow f such that $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

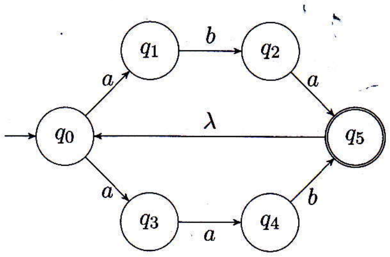
(c) A minimal (s,t)-cut in G is unique if all capacities c_a are pairwise distinct.

(d) Multiplying each of the capacities c_a by one and the same number $\lambda > 0$ does not change the minimal (s,t)-cuts.

(e) Adding one and the same number $\lambda > 0$ to each of the capacities c_a does not change the minimal (s,t)-cuts.

Languages & Machines

9. *(10 points)* Consider the following NFA-λ, M (only q_5 is accepting):



(a) Provide the λ-closure and input-transition function of M in a table.

(b) Transform the automaton M systematically to an incomplete DFA.

(c) Construct systematically a regular expression E with $\mathcal{L}(E) = \mathcal{L}(M)$.

10. *(10 points)* We introduce the following three languages:

- language $L_1 := \{a^i\, b^j\, c^k \mid i > 0 \text{ and } 0 \le j \le k\}$
- language $L_2 := \{a^i\, b^j\, b^k \mid i > 0 \text{ and } 0 \le j \le k\}$
- L_3 is an (abritrary) *finite* language.

Indicate whether the following languages are regular or not. Prove your answers.

(a) The language L_1.

(b) The language $L_2 \cup L_3$.