Re-Exam 1, Module 7, Code 201600270

Discrete Structures & Efficient Algorithms

Tuesday, April 18, 2017, 08:45 - 11:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4) per topic (ADS, DM, L&M).

This exam consists of three parts, with the following (estimated) time requirements:

Algorithms & Data Structures (ADS)	1h	(30 points)
Discrete Mathematics (DW)	1h 20 min	(40 points)
Languages & Machines (L&M)	40 min	(20 points)

Total 30+40+20=90 points. Your exam grade is the total number of points plus 10, divided by 10.

Please use a new sheet of paper for each part (ADS/DM/L&M)!

Algorithms & Data Structures

1. (10 points) Given an integer array a with length $n = 2^k$ for some $k > 0$. Consider the following algorithm for determining the maximum and the minimum:

```
(int, int) minmax( int a[1], ..., a[n])
{ if (n==2)
   { if (a[1]<=a[2]) return (a[1],a[2]); else return (a[2],a[1])
   }
  else { (mn1,mx1) = minmax(a[1], ..., a[n/2]);
         (mn2,mx2) = minmax(a[n/2+1], ..., a[n]);
         return(min(mn1,mn2), max(mx1,mx2));
       }
}
```

   (a) Give a recursive expression for the number of comparisons of this algorithm.

   (b) Give the asymptotic time complexity of this algorithm.

2. (10 points)

   (a) Give an efficient algorithm that deletes the maximum of a maxheap (the result should again be a maxheap). What is de time complexity of your algorithm?

   (b) Given a nonempty binary search tree with unique elements. Give an algorithm that determines the biggest element smaller that the maximum element (and explain your solution).

3. (10 points) A mechanic has a list of n jobs he has to perform. He has made an estimate of the time t_i (an integer) he needs to finish each job i. He wants to work tomorrow at least T minutes: certainly not less, maybe a bit longer, but preferably as little longer as possible.

   (a) The function $B(i, t)$ indicates the amount of extra time for an optimal choice of jobs from $k_i, \ldots, k_n$ if you want to work for at least t minutes. Explain that

- $B(i,0) = 0$ for $1 \leq i \leq n+1$
- $B(n+1,t) = \infty$ if $t > 0$
- $B(i,t) = min(t_i - t, B(i+1,t))$ if $t_i \geq t$
- $B(i,t) = min(B(i+1, t - t_i), B(i+1,t))$ otherwise

(b) Give an algorithm that determines the minimal amount of extra time if you want to work for T mintes. Use dynamic programming, based on the equations for function $B(i,t)$.

Discrete Mathematics

4. (5 points) Show that the Diophantine equation $236s + 24t = 2$ has no solution for $s, t \in \mathbb{Z}$.

5. (10 points)

   (a) Let us denote by a_n the number of strings in $\{a, b, c\}^*$ of length n that contain an even number of a's. Compute a_1, a_2, and give a recurrence relation for a_n, $n \geq 3$. (You do not have to solve the recurrence relation.)

   (b) Compute the solution to the recurrence relation

   $$a_n - 6a_{n-1} + 9a_{n-2} = 4n + 4 \quad (n \geq 2) \quad \text{with } a_0 = 5 \text{ and } a_1 = 9\,.$$

6. (8 points) Suppose we are given a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of (directed) arcs, and $c_a \geq 0$, $a \in A$ are the arc capacities. Also, let $s, t \in V$ and $f = (f_a)_{a \in A}$ be a feasible flow in G. Give a short proof or give a counterexample for each of the following statements.

   (a) There is a maximal (s,t)-flow f such that $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

   (b) A minimal (s,t)-cut in G is unique if all capacities c_a are pairwise distinct.

   (c) Multiplying each of the capacities c_a by one and the same number $\lambda > 0$ does not change the minimal (s,t)-cuts.

   (d) Adding one and the same number $\lambda > 0$ to each of the capacities c_a does not change the minimal (s,t)-cuts.

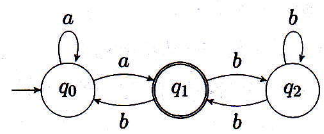
7. (5 points) Suppose you are given an undirected (simple) graph $G = (V, E)$ with $|E| = 35$, and $d(v) \geq 5$ for all $v \in V$. How many nodes can the graph possibly have? (Give both a min. and a max.)

8. (7 points) Let $G = (V, E)$ be a simple, undirected graph with edge lengths $d_e \geq 0$, $e \in E$. Let $T \subseteq E$ be an arbitrary minimal spanning tree (MST) for G. Also, for a given $s \in V$ let D_s be the union of *all* shortest (s, v)-paths, for all $v \in V \setminus \{s\}$. Show that $T \cap D_s \neq \emptyset$.

9. (5 points) How many possibilities are there to select six nonconsecutive numbers from the set $\{1, 2, \ldots, 50\}$? Use a generating function.

Languages & Machines

10. (10 points) Consider the following NFA, M (only q_1 is accepting):



(a) Provide the input-transition function of M in a table.

(b) Transform the automaton M systematically into an (incomplete) DFA.

(c) Construct systematically a regular expression E with $\mathcal{L}(E) = \mathcal{L}(M)$.

11. (10 points) Consider the following languages:

- language $L_1 := \{a^k\, b^{2j} \mid k \geq j \geq 0\}$
- language $L_2 := \{a^k\, b^{2j} \mid j \geq k \geq 0\}$

Indicate whether the following languages are regular or not. Prove your answers.

(a) The language $L_1 \cup L_2$ (union).

(b) The language $L_1 \cap L_2$ (intersection).