Kenmerk: EWI2018/TW/DMMP/MU/Mod7/Exam1

Exam 1, Module 7, Code 201700304

Discrete Structures & Efficient Algorithms

Monday, March 19, 2018, 08:45 - 11:45

All answers need to be motivated. No calculators. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM, L&M).

This exam consists of three parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	1h	(30 points)
Discrete Mathematics (DW)	1h 20 min	(40 points)
Languages & Machines (L&M)	40 min	(20 points)

The total is 30+40+20=90 points. Your exam grade is the maximum of 1 and the total number of points divided by 9, rounded to one digit.

Important: It is necessary to use a new sheet of paper for each part (ADS/DW/L&M)!

Algorithms & Data Structures

1. (10 points) Consider the following algorithm (where * is multiplication, // integer division (eg. 7 // 2 = 3), and ** is exponentiation:

```
def func(n):
    if n==0:
        return 1
    else:
        if n<4:
            return n
        else:
            return 3*func(n//4) + 6 + func(n//4)**2
```

   (a) Give a recursive expression for the time complexity $T(n)$ of this algorithm (measured in the number of arithmetic operations).

   (b) Use the recursive expression for determining the asymptotic complexity.

2. (5 points) Given a maxheap E with n elements. Give an algorithm that returns the difference between the maximum and the minimum in this heap. The algorithm should make no more than $n/2$ comparisons.

3. (15 points) A game is played in a garden divided into n times n squares. In each square there is a number of pearls (at least 1 per square). You start in the leftmost lowest square (that has coordinates $(1, 1)$) and you are allowed each time to move upward or to the right. You have to end in the rightmost highest square (that has coordinates (n, n)). You may take the pearls in each square that you pass. The goal is to get as much pearls as possible.

(a) Let $c(i,j)$ be the number of pearls in square (i,j), and $P(i,j)$ be the optimal total number of pearls you have collected when you arrive in square (i,j). Assume $P(i,j) = 0$ for $i = 0$ or $j = 0$. Motivate which of the following recurrence relations hold for $1 \leq i \leq n$, $1 \leq j \leq n$:

   i. $P(i,j) = max\{P(i-1, j + c(i,i)), P(i-1,j)\}$
   ii. $P(i,j) = c(i,j) + max\{P(i-1,j), P(i,j-1)\}$
   iii. $P(i,j) = c(i,j) + min\{P(i,j-1), P(i-1,j)\}$
   iv. $P(i,j) = max\{P(i-1, j + c(i,j)), P(i + c(i,j), j-1)\}$

(b) Give an algorithm to determine the maximal number of pearls you can win. The complexity may not be bigger than $\Theta(n^2)$.

Discrete Mathematics

4. (5 points) Suppose we only had two different types of bills with face values a€ and b€, $a, b \in \mathbb{Z}$. Somebody tells you that he had recently paid 64€. Another person tells that she had paid 45€. Show that, for any $k \in \mathbb{Z}$, the amount k€ can be paid using only bills of a€ and b€.

5. (10 points)

   (a) Denote by a_n the number of strings in $\{0, 1, 2\}^*$ of length n that contain no substring 11, and neither 22. Compute a_1 and a_2. Set up a recurrence relation for a_n, $n \geq 3$. (You do not have to solve that recurrence relation.)

   (b) Compute the solution to the recurrence relation

$$a_n - 10a_{n-1} + 25a_{n-2} = 16n + 8 \quad (n \geq 2) \quad \text{with } a_0 = 3 \text{ and } a_1 = 12\,.$$

6. (8 points) Suppose we want to share 100€ among three persons, such that each of them gets at least 20€, but at most 50€, and moreover each person gets an integer amount. How many possibilities are there to do that? Use a generating function to compute your answer.

7. (7 points) Let $G = (V, E)$ simple, connected, undirected graph with $|V| = n$ and $|E| = m$. Show that, if G has an average node degree $\sum_{v \in V} \frac{d(v)}{n} \geq 6$, then G cannot be planar.

8. (10 points) Give a short proof or give a counterexample for each of the following statements.

   (a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$. Then any two minimum spanning trees T_1 and T_2 for G must have a nonempty intersection, that is, $T_1 \cap T_2 \neq \emptyset$.

   (b) Consider a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, and $c_a \geq 0$, $a \in A$ are the arc capacities. Then there always exists a maximum flow f_a, $a \in A$, such that either $f_a = 0$ or $f_a = c_a$ for all $a \in A$.

   (c) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Let $s \in V$ be fixed. Then for all $v \in V$ there is a unique shortest (s, v)-path.

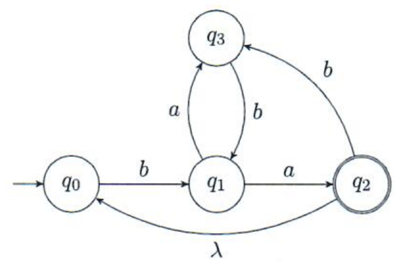
   (d) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$ such that $w_e \neq w_{e'}$ for all $e, e' \in E$, $e \neq e'$. Then there is a unique minimum spanning tree T.

(e) In a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, $c_a \geq 0$, $a \in A$, are the arc capacities, and $c_a \neq c_{a'}$ for all $a, a' \in A$, $a \neq a'$. Then there is a unique minimum cut.

Languages & Machines

Please remember to start this part on a new sheet of paper!

9. (11 points) Consider the following NFA with λ-steps M (only q_2 is accepting):



(a) First eliminate state q_3, adding new transitions labeled with regular expressions, to preserve the accepted language. Show the resulting "expression graph".

(b) Continue the construction, by eliminating q_1 as well, and read off a regular expression E with $\mathcal{L}(E) = \mathcal{L}(M)$.

(c) Provide the λ-closure and input-transition function of the automaton M in a table.

(d) Transform the automaton M in a systematic manner to a (possibly incomplete) DFA.

10. (9 points) Consider the definitions of the following languages over $\Sigma = \{a, b\}$:

- Language $L_1 := \{a^{2i} b^j \mid 0 \leq i \text{ and } 0 \leq j\}$
- Language $L_2 := \{a^i b^j a^i \mid 0 \leq i \text{ and } 0 \leq j\}$
- Language L_3 is an (arbitrary) *finite* language

Indicate for each of the following languages if they are regular or not. Motivate your answers, either by a proof or a construction.

(a) Language L_1

(b) Language L_2

(c) Language $\overline{L_3} \cup L_3^R$