Kenmerk: EWI2020/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 201400483 & 201800141

Discrete Structures & Efficient Algorithms

Friday, March 27, 2020, 08:45 - 11:45

All answers need to be motivated, arguments and proofs must be complete, and reference to online sources is not sufficient. You are allowed to use the textbook(s), lecture slides, as well as your handwritten cheat sheet per topic (ADS, DM) during the exam.

This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please read carefully: By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test. In case of doubt, it might be that we have to decide not to count the test result, which could include invalidating the test results of all other students, too. Therefore, our appeal is to your own responsibility:

You maximise your own, and all your fellow students' chance to have this test result remain valid, by adhering to the rules as stated below.

In order for the test to be graded, the following text must be copied on the first page of your solutions:

> **"I have made this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test" [Name, Student no., Location, Date, Signature].**

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will have to be uploaded separately!

Algorithms & Data Structures

1. (10 points) Given a sequence of numbers $a_1, \dots, a_n$ of length $n = 2^k$ for some $k > 0$. Consider the following algorithm (in pseudocode) for determining the minimum and maximum of the sequence:

```
minmax( a[1], ..., a[n])
{ if (n==2)
   { if (a[1]<=a[2]) return (a[1],a[2]); else return (a[2],a[1])
   }
```

```
    else { (mn1,mx1) = minmax(a[1], ..., a[n/2]);
           (mn2,mx2) = minmax(a[n/2+1], ..., a[n]);
           return(min(mn1,mn2), max(mx1,mx2));
         }
}
```

(a) Give a recurrence relation for the time complexity of this algorithm, expressed in the number of comparisons.

(b) What is the asymptotic complexity of this algorithm?

2. (a) (5 points) Given a maxheap E with n elements. Give an algorithm that returns the difference between the maximum and the minimum in this heap. The algorithm should make no more than $n/2$ comparisons.

   (b) (5 points) Given a completely filled binary tree of depth 3, where each node has as a (unique) key one of the letters A, B, ..., or O, in such a way that the tree is alphabetically sorted in–order.

   What is the order in which you encounter the letters if you traverse the tree in a pre–order way?

3. (10 points) Given a backpack with maximum weight capacity G. Given n objects 1 to n where each object i has weight w_i. Suppose all weights are integers. The goal is to fill the backpack with objects, with as much weight as possible.

   (a) Suppose at a certain point you are considering objects 1 to i, and you still have weight g available in the backpack. Define $R(i,g)$ as the remaining (unused) weight of the backpack if you pack as much weight as possible adding objects from 1 to i.

   Motivate which of the following recurrence relations holds:

   - $R(i,g) = w_i + max\{R(i-1,g), R(i,g-1)\}$
   - $R(i,g) = min\{R(i-1,g), R(i-1,g-w_i)\}$
   - $R(i,g) = w_i + min\{R(i-1,g), R(i,g)\}$
   - $R(i,g) = min\{R(i-1,g), R(i,g-w_i)\}$

   (b) Give an algorithm to determine the maximum weight that can be put into the backpack. The complexity may be no worse that quadratic in n.

Discrete Mathematics

4. (10 points)

   (a) For given and fixed integer numbers $a, b \in \mathbb{Z}$, assume that we know that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ so that $\alpha a + \beta b = 165$ and $\gamma a + \delta b = 98$. Use the Euclidean algorithm and show that $\gcd(165, 98) = 1$ (give all iterations until termination). Use this result to prove that a and b are relatively prime.

   (b) Suppose you live in the United States of Number Theorists, a country that only features €-bills with face values 165€ and 98€. You want to pay somebody an amount of 5€. Is this possible? If yes, how can the transaction of 5€ be done?

5. (10 points)

   (a) Denote by a_n the number of strings in $\{0,1,2\}^*$ of length n that do not contain consecutive 0's. Compute a_1 and a_2, and set up a recurrence relation for a_n, for all $n \geq 3$. You do not need to solve this recurrence relation, but your need to give a proof why your recurrence relation is correct.

   (b) Compute the solution to the following recurrence relation.

$$a_{n+2} = 3a_{n+1} + 10a_n + 7 \cdot 5^n, \text{ with } a_0 = 0 \text{ and } a_1 = 15\,.$$

6. (10 points)

   (a) Suppose we want to donate 100€ Euros to three different charity organizations C_1, C_2 and C_3, such that each each of the three gets at least 20€, and at most 50€. How many possibilities are there to do that? Use a generating function to compute your answer.

   (b) If the question is to split 100€ into three parts, such that each part is an integer amount of at least 20€ and at most 50€, is the answer

   - smaller than
   - equal to
   - larger than

   the answer in (a)? Explain why.

7. Let $G = (V, E)$ be a simple undirected graph with $|V| = n$ and $|E| = m$. For each of the following two statements, decide wether it is true or not. Either give a short proof or a counterexample.

   (a) (4 points) If G is planar, then $m \leq 3n - 6$.

   (b) (4 points) If $m \leq 3n - 6$, then G is planar.

   Now let G be a simple undirected *bipartite* graph with $|V| = n$ and $|E| = m$.

   (c) (6 points) If the minimum vertex degree $\min\{d(v) \mid v \in V\} \geq 4$, prove that G cannot be planar.

8. (16 points – 4 points each) For each of the following four claims, decide wether it is true or not. Either give a short proof or a counterexample.

   (a) Consider a directed, simple graph $G = (V, E)$ with $s, t \in V$ fixed, so that t is reachable from s, and with edge lengths $\ell(e) \geq 0$ such that $\ell(e) \neq \ell(e')$ for all $e \neq e'$. **Claim:** For any two shortest (s,t)-paths $P_1, P_2 \subseteq E$, we must have $P_1 \cap P_2 \neq \emptyset$.

   (b) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w(e) \geq 0$, $e \in E$, and $w(e) \neq w(e')$ for all $e \neq e' \in E$. **Claim:** For any two minimum spanning trees $T_1, T_2 \subseteq E$, we must have $T_1 \cap T_2 \neq \emptyset$.

   (c) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are edge capacities. **Claim:** Let f be some (s,t)-flow in G respecting the flow balance and capacity constraints, and let $\text{val}(f)$ be its value. Then there exists an (s,t)-cut $[S,T]$ with the same capacity, $\text{cap}[S,T] = \text{val}(f)$.

   (d) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are edge capacities. **Claim:** If there exists one edge e which is part of *every* minimum capacity (s,t)-cut $[S,T]$, then in *every* maximum flow f, we have $f(e) = c(e)$.