Kenmerk: EWI2021/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 202001360 & 202001364

Discrete Structures & Efficient Algorithms

Friday, March 19, 2021, 09:00 - 12:00

All answers need to be motivated, arguments and proofs must be complete. You are allowed to use your cheat sheets during the exam (2 A4, both sides).

Important: When finished, and before you hand in your exam, you must take pictures of all your solutions using your mobile phone. These solutions must be uploaded to the Canvas site (there is an assignment for that) *in one single pdf*, before 14:00 on the same day.

For information: This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will be corrected separately!

Double students Discrete Mathematics & Algebra (202001364) only do the DM part. In that case, please write "Discrete Mathematics & Algebra" on top of your exam.

Algorithms & Data Structures

1. (10 points)

   (a) Consider this sorting algorithm that sorts an array *arr* of integers:

   ```
   def Sort(arr):
       n = len(arr)

       for i in range(n-1):
           for j in range(0, n-i-1):

               if arr[j] > arr[j+1] :
                   arr[j], arr[j+1] = arr[j+1], arr[j]
   ```

   Give the asymptotic time complexity of this algorithm, expressed in the number of comparisons. Is this an in–place algorithm?

(b) Give the asymptotic order of the solution of the following recurrence equation:

$$T(n) = 8T(n/2) + n^3 + 4n + 1/n$$

2. (a) (5 points)
Given an array A sorted in decreasing order. Give an efficient algorithm that turns A into a heap.

(b) (5 points)
Given a completely filled binary tree of depth 3, where each node has as a (unique) key one of the letters A, B, ..., or O, in such a way that the tree is alphabetically sorted post–order.
What is the order in which you encounter the letters if you traverse this tree in a pre–order way?

3. (10 points)
A game is played in a garden divided into n times n squares. In each square there is a number of pearls (at least 1 per square). You start in the leftmost lowest square (that has coordinates $(1, 1)$) and you are allowed each time to move upward or to the right. You have to end in the rightmost highest square (that has coordinates (n, n)). You may take the pearls in each square that you pass. The goal is to get as much pearls as possible.

(a) Let $c(i, j)$ be the number of pearls in square (i, j), and $P(i, j)$ be the optimal total number of pearls you have collected when you arrive in square (i, j). Assume $P(i, j) = 0$ for $i = 0$ or $j = 0$. Motivate which of the following recurrence relations hold for $1 \leq i \leq n$, $1 \leq j \leq n$:

i. $P(i, j) = max\{P(i-1, j + c(i, i)), P(i-1, j)\}$
ii. $P(i, j) = c(i, j) + max\{P(i-1, j), P(i, j-1)\}$
iii. $P(i, j) = c(i, j) + min\{P(i, j-1), P(i-1, j)\}$
iv. $P(i, j) = max\{P(i-1, j + c(i, j)), P(i + c(i, j), j-1)\}$

(b) Give an algorithm to determine the maximal number of pearls you can win. The complexity may not be bigger than $\Theta(n^2)$.

Discrete Mathematics

4. (10 points)

(a) Consider the equation

$$5 = 385x + 150y\,.$$

Use the extended Euclidean algorithm to either compute an integer solution $x, y \in \mathbb{Z}$, or show that no such solution is possible.

(b) Let $a, b \in \mathbb{Z}$ be arbitrary, not both equal to 0, and let $k > 0$ be an integer. Give a proof or a counterexample:

i. $\gcd(a + k, b + k) = \gcd(a, b) + k$.

ii. $\text{gcd}(a \cdot k, b \cdot k) = \text{gcd}(a, b) \cdot k$.

5. (6 points)

   Compute the solution to the following recurrence relation. Use either the characteristic polynomial, or a generating function.

   $$a_{n+2} = 4a_{n+1} - 4a_n + n, \text{ with } a_0 = 4 \text{ and } a_1 = 17\,.$$

6. We are given a simple, directed graph $G = (V, E)$ with edge lengths $c(u, v) \geq 0$ for all $(u, v) \in E$, and $s \in V$ so that there exists a directed (s, v)-path for all $v \in V$. Now consider the following system of equations in variables x_v, $v \in V$:

   $$x_s = 0$$
   $$x_v = \min\{x_u + c(u, v) \mid (u, v) \in E\} \quad \forall\, v \neq s\,.$$

   (a) (5 points) Let us *define* x_v as the length of a shortest (s, v)-path, for all $v \in V$. Show that *then*, $x = (x_v)_{v \in V}$ is a solution to the above system of equations.

   (b) (5 points) Let us *define* $x = (x_v)_{v \in V}$ as any solution to the above system of equations. Show that *then*, x_v *need not* be equal to the length of a shortest (s, v)-path, for all $v \in V$.

7. Suppose we want to donate 300€ Euros to three different NGO's NGO_1, NGO_2 and NGO_3, such that each NGO gets an integer amount of €'s, at least 50€ but no more than 150€.

   (a) (7 points) How many possibilities are there to do that? Use a generating function to compute your answer.

   (b) (3 points) If we wonder in how many ways we can split 300€ into three parts, such that each part is an integer amount of at least 50€ and at most 150€, is the answer

   - smaller than
   - equal to
   - larger than

   the answer in (a)? Explain (in one sentence) why.

8. (6 points) Let $G = (V, E)$ be a simple, undirected and connected graph with $|V| = n \geq 12$ vertices and $|E| = m$. Assume that at least half of the vertices of G have a degree $d(v)$ at least 11. Prove that G cannot be planar.

9. (3 points each) For each of the following six claims, decide whether it is true or false. **As your answer, only write "true" or "false"**. But note: **A correct answer gives three points, an incorrect answer gives minus two points**. The total number of points for this question is $\max\{0, 3x - 2y\}$ where x is the number of correct, and y is the number of incorrect answers.

   (a) Consider an undirected graph $G = (V, E)$ with edge weights $w(e) \geq 0$, $e \in E$. **Claim:** If T^* is the edge set of any minimum spanning tree for G, then T^* also minimizes the function $f(T) := \max\{w(e) \mid e \in T\}$ among the set of all spanning trees T of G.

   (b) Consider an undirected graph $G = (V, E)$ with edge lengths $\ell(e) \geq 0$, $e \in E$. **Claim:** If S^* is the edge set of any shortest path tree[1] of G, then S^* also minimizes the function $f(S) := \max\{\ell(e) \mid e \in S\}$ among the set of all shortest path trees S of G.

[1] That is the set of edges of, say, one application of Dijkstra's shortest path algorithm.

(c) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are the integer edge capacities. **Claim:** For any feasible (s,t)-flow f in G, there exists an (s,t)-cut $[S, T]$ with capacity equal to f's flow value.

(d) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are the integer edge capacities. **Claim:** If f is a given maximum flow for G, and we increase the capacity of only one edge of G by some integer amount k, then we can compute in $O(|V| + |E|)$ time a maximum flow f^{new} in the new network.

(e) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are the integer edge capacities. **Claim:** If f is a given maximum flow for G, then we can compute in $O(|V| + |E|)$ time a minimum capacity cut in G.

(f) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are edge capacities. **Claim:** When adding one and the same integer $k > 0$ to all of the capacities $c(e)$, we still have the same set of minimal (s,t)-cuts (only with larger total capacity).