Kenmerk: EWI2021/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 202001360 & 202001364

Discrete Structures & Efficient Algorithms

Friday, March 25, 2022, 13:45 - 16:45

Answers to questions 1-8 need to be motivated, arguments and proofs must be complete. You are allowed to use a handwritten cheat sheet (one A4, both sides) per topic (ADS, DM) during the exam.

For information: This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will be corrected separately!

Double students Discrete Mathematics & Algebra (202001364) only do the DM part. In that case, please write "Discrete Mathematics & Algebra" on top of your exam.

Algorithms & Data Structures

1. (10 points)

   (a) Consider this sorting algorithm that sorts an array *arr* of integers:

   ```
   def Sort(arr):
       n = len(arr)

       for i in range(n-1):
           for j in range(0, n-i-1):

               if arr[j] > arr[j+1] :
                   arr[j], arr[j+1] = arr[j+1], arr[j]
   ```

   Give the asymptotic time complexity of this algorithm, expressed in the number of comparisons. Is this an in–place algorithm?

   (b) Give the asymptotic order of the solution of the following recurrence equation:

   $$T(n) = 4{\cdot}T(n/2) + n^3$$

2. (10 points)

   (a) Turn the following array into a maxheap, and explain your steps:

18	16	12	10	7	9	1	8	6	3

   (b) Give a completely filled binary tree of depth 3, where each node has as a (unique) key one of the numbers from 1 to 15, in such a way that the tree is sorted in–order.

   What is the order in which you encounter the numbers if you traverse this tree in an pre–order way?

3. (10 points)

   Given a backpack with maximum weight capacity G. Given n objects 1 to n where each object i has weight w_i. Suppose all weights are integers. The goal is to fill the backpack with objects, with as much weight as possible.

   (a) Suppose at a certain point you are considering objects 1 to i, and you still have weight g available in the backpack. Define $R(i, g)$ as the remaining (unused) weight of the backpack if you have packed as much weight as possible adding objects from 1 to i.

   Motivate which of the following recurrence relations holds:

   i. $R(i, g) = w_i + max\{R(i-1, g), R(i, g-1)\}$
   ii. $R(i, g) = min\{R(i-1, g), R(i-1, g-w_i)\}$
   iii. $R(i, g) = w_i + min\{R(i-1, g), R(i, g)\}$
   iv. $R(i, g) = min\{R(i-1, g), R(i, g-w_i)\}$

   (b) Give an algorithm to determine the maximum weight that can be put into the backpack. The complexity may be no worse that quadratic in n.

Reminder: Please use a new sheet of paper for each part (ADS, DM), as the ADS and the DM parts will be corrected separately!

Discrete Mathematics

4. $(5 + 3 + 2$ points)

   (a) Consider the Diophantine equation

$$30 = 141x + 34y.$$

   Use the extended Euclidean algorithm to either compute an integer solution $x, y \in \mathbb{Z}$, or show that no such solution is possible.

   (b) Consider $a, b, c \in \mathbb{Z}$ with $\gcd(a, b) = 1$ and $a|bc$. Prove that $a|c$.

   (c) Prove that $\gcd(F_n, F_{n-1}) = 1$ where F_n is the n'th Fibonacci number.[1]
   (Hint: You may want to use mathematical induction.)

[1] *Reminder:* The Fibonacci numbers are defined by the recurrence relation $F_0 = 0$, $F_1 = 1$, and for all $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$.

5. (7 points) Prove that any tree $T = (V, E)$ with maximum degree k must have at least k vertices with degree 1.
(Hint: You may use the fact that every tree $T = (V, E)$ with $|V| \geq 2$, has at least two vertices of degree 1.)

6. (7 points) Consider a graph $G = (V, E)$ with edge costs $c(e), e \in E$ such that all edge costs are distinct (i.e., for any $e, e' \in E, c(e) = c(e')$ implies that $e = e'$). Prove that in this case there exists a *unique* minimum spanning tree T in G.
(Hint: Proof by contradiction.)

7. (7 points) Consider a simple, capacitated network $G = (V, E, c)$, where V is the set of vertices, $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$ for $e \in E$ are the edge capacities. Let $|V| = n$ and $|E| = m$. Suppose you are given a maximum (s, t)-flow f. Suggest how to compute a minimum (s, t)-cut $[S, T]$ for G in time $O(n + m)$. Briefly explain (i) why your suggested algorithm is correct (ii) why it achieves the desired running time.
(Hint: You may want to use the residual graph for G and the maximum (s, t)-flow.)

8. (3+4+4 points)

   (a) There are n seating positions placed in a line. Consider the seating arrangements (i.e., subsets of the n seating positions) that do not contain any consecutive seats. Prove that the number of such arrangements is F_{n+1}, where F_n is the n'th Fibonacci number.

   (b) Give a recurrence relation for the number of seating arrangements if the n seating positions are placed around a circle, $n \geq 3$, and again consecutive seats are disallowed. Explain why your recurrence relation is correct.

   (c) Compute the solution to the recurrence relation

$$a_{n+2} - 7a_{n+1} + 10 = 0, \quad n \geq 0, \quad \text{with } a_0 = 0 \text{ and } a_1 = 3.$$

9. (3 points each) For each of the following six claims, decide if true or false or if you would rather not give an answer. A correct answer gives **3**, an incorrect answer **−2** and not giving an answer **0 points** (minimum total number of points for Question 9 is 0 points). **Instead of guessing, it may be better not giving an answer.**

   (a) The complete bipartite graph $K_{m,n}$ contains an Euler tour if and only if both m and n are even.

   True .... ☐
   False.................................. ☐
   I prefer to not give an answer .... ☐

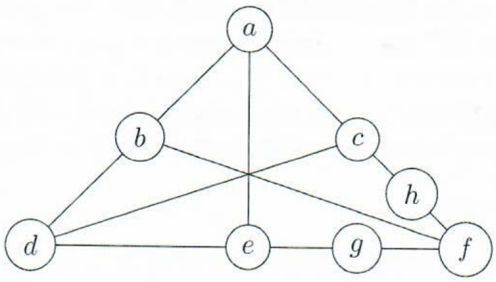
   (b) Consider a graph $G = (V, E)$ and a minimum spanning tree T on G. Then for every pair of vertices $u, w \in V$, the shortest (u, w)-path in G is contained in T.

   True .... ☐
   False.................................. ☐
   I prefer to not give an answer .... ☐

(c) Consider a capacitated network $G = (V, E, c)$, where $s, t \in V$, E is the set of directed edges, and $c(e) \geq 0$, $e \in E$, are the integer edge capacities. When adding the same integer $k > 0$ to all the capacities $c(e)$, we still have the same set of minimal (s, t)-cuts (only with larger total capacity).

True . . . . ☐
False. . . . . ☐
I prefer to not give an answer . . . . ☐

(d) The following graph G is planar.



True . . . . ☐
False. . . . . ☐
I prefer to not give an answer . . . . ☐

(e) The generating function for the sequence $1, 0, 1, 0, 1, 0, 1, \ldots$ is

$$f(x) = \frac{1}{1 - x^2}.$$

True . . . . ☐
False. . . . . ☐
I prefer to not give an answer . . . . ☐

(f) The coefficient of x^{50} in $(x^7 + x^8 + x^9 + \ldots)^6$ is $\binom{14}{8}$.

True . . . . ☐
False. . . . . ☐
I prefer to not give an answer . . . . ☐