Kenmerk: EWI2021/TW/MOR/MU/Mod7/Exam1

Exam 1, Module 7, Codes 202001360 & 202001364
Discrete Structures & Efficient Algorithms
Monday, March 25, 2022, 8:45 - 11:45

Answers to questions 1-8 need to be motivated, arguments and proofs must be complete. You are allowed to use a handwritten cheat sheet (A4, both sides) per topic (ADS, DM) during the exam.

For information: This exam consists of two parts, with the following (estimated) times per part:

Algorithms & Data Structures (ADS)	ca. 1h	(30 points)
Discrete Mathematics (DM)	ca. 2h	(60 points)

The total is 30+60=90 points. Your grade is $1 + 0.1x$, x being the number of points, rounded to one digit. That means, you need 45 points to get a 5.5.

Please use a new sheet of paper for each part (ADS, DM), as the ADS and DM parts will be corrected separately!

Double students Discrete Mathematics & Algebra (202001364) only do the DM part. In that case, please write "Discrete Mathematics & Algebra" on top of your exam.

Algorithms & Data Structures

1. (10 points)

   (a) Examine the following algorithm:

   ```
   def func(n):
       res=0

       while n>0:
           m=n
           while m>0:
               res=res+m
               m=m-1
           n=n-1

       return res
   ```

   Give the asymptotic time complexity of this algorithm expressed in the number of arithmetic operations.

   (b) Suppose that the number of steps of an algorithm $T(n)$ with an input of n, has the recurrence relation

   $$T(n) = 8T(n/2) + n^2 + 4n + 1/n$$

   What is the asymptotic complexity class of this algorithm?

2. (10 points)

   (a) Give an algorithm that deletes the smallest element in a minheap, returning a heap with the remaining elements. The complexity of this algorithm must be $O(log\ n)$.

   (b) Given a non-empty binary tree sorted post-order. Give an algorithm that yields the node with the biggest element and the node with the smallest element.

3. (10 points)

   Given n positive integers $a_1, \ldots, a_n$ and positive integer G. We want to determine whether there is a subsequence of $a_1, \ldots, a_n$ with sum G. Example: for $7, 3, 2, 5, 8$ and $G = 14$ the answer is "true" since $7 + 2 + 5 = 14$.

   Define boolean $R(i, g)$ as:
   $R(i, g)$ iff $a_1, \ldots, a_i$ has subsequence with sum g.

   Note that

   - $R(i, 0) = true$ for all $0 \leq i \leq n$

- $R(0, g) = false$ for all $g > 0$
- $R(i, g) = false$ for all $g < 0$

(a) Motivate which of the following recurrence relations holds (for all $1 \leq i \leq n$, $1 \leq g \leq G$):

i. $R(i, g) = (R(i-1, g) \leq a_i + R(i, g))$

ii. $R(i, g) = R(i-1, g)||R(i-1, g-a_i)$

iii. $R(i, g) = min\{R(i+1, g), R(i+1, g+a_i)\}$

iv. $R(i, g) = R(i-1, g)||R(i, g-a_i)R(i-1, g)$

(b) Give an algorithm to determine whether $a_1, \ldots, a_n$ has a subsequence with sum G. The complexity may be no worse than $\Theta(Gn)$.

Reminder: Use a new sheet of paper for each part (ADS, DM), as the ADS and the DM parts will be corrected separately!

Discrete Mathematics

4. (4+5 points)

(a) Consider $a \in \mathbb{Z}$, with $a > 1$ and its (unique) prime factorization

$$a = p_1^{a_1} \cdot p_2^{a_2} \ldots p_k^{a_k}.$$

Show that a is a perfect square[1] if and only if for all $1 \leq i \leq k$, a_i is even.

(b) Consider $a, b \in \mathbb{Z}_{>0}$, with $\gcd(a, b) = 1$. Show that if ab is a perfect square then both a and b must be perfect quares.
[Hint: You may want to use the statement of (a).]

5. (8 points) Consider a bipartite graph $G = ((V_1, V_2), E)$ for which every vertex has the same positive degree. In other words, there exists an integer $\Delta \geq 1$ such that $deg(v) = \Delta$ for all $v \in (V_1 \cup V_2)$. Prove that $|V_1| = |V_2|$.

6. (10 points) Consider a simple, capacitated network $G = (V, E, c)$, where V is the set of vertices, $s, t \in V$, E is the set of directed edges, and $c : E \rightarrow \mathbb{Z}_{\geq 0}$ gives the edge capacities. Let $|V| = n$ and $|E| = m$. Suppose you are given a maximum (s, t)-flow f. Now suppose that the capacity of a specific edge $e \in E$ is increased by one unit, and let G' be the resulting capacitated network. Suggest how to compute the maximum (s, t)-flow in f' in G', in $O(n+m)$ time. Briefly explain (i) why your suggested algorithm

[1] An $x \in \mathbb{Z}$ is a *perfect square* if and only if there exists $y \in \mathbb{Z}$ such that $x = y^2$.

is correct (ii) why it achieves the desired running time.
[Hint: You may want to use the (s,t)-flow f as well as the residual graph for G' with respect to f.]

7. (4+3 points)

   (a) Compute the solution to the recurrence relation

   $$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n \quad (n \geq 2) \quad \text{with} \quad a_0 = 1 \text{ and } a_1 = 2\,.$$

   (b) Let a_n with $n \geq 0$ be the number of bit strings (i.e., strings in $\{0,1\}$) of length n that contain three consecutive zeros. Find a recurrence relation for computing a_n. (You do not need to solve this recurrence relation.)

8. (8 points) Assume that Alice has published modulus $n = 55$, and exponent $e = 7$. Bob sends ciphertext $C = 5$ to Alice. You are eavesdropper Eve and you are interested in Bob's secret message M. Compute Bob's secret message M from ciphertext C. Write down all of the computational steps that you need to perform in order to obtain Bob's secret message M.

9. (3 points each) For each of the following six claims, decide if true or false or if you would rather not give an answer. A correct answer gives **3**, an incorrect answer **−3** and not giving an answer **0 points** (minimum total number of points for Question 9 is 0 points). **Instead of guessing, it may be better not giving an answer.**

   (a) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w_e \geq 0$, $e \in E$. Consider a cycle C in G, and let e be an edge of the cycle C such that $w_e \leq w_{e'}$ for any edge e' along C. Then e must belong to a minimum spanning tree of G.

   True . . . . ☐
   False . . . . ☐
   I prefer to not give an answer . . . . ☐

   (b) Consider a capacitated network $G = (V, A, c)$, where V is the set of vertices, A is the set of directed arcs, and $c : A \to \mathbb{Z}_{\geq}0$, are the arc capacities. Let f be some (s,t)-flow in G respecting the flow conservation and capacity constraints, and let $val(f)$ be its value. Then there must exist an (s,t)-cut $[S,T]$ with capacity $cap([S,T]) = val(f)$.

   True . . . . ☐
   False . . . . ☐
   I prefer to not give an answer . . . . ☐

(c) Consider an undirected, simple graph $G = (V, E)$ with edge weights $w : E \to \mathbb{R}$, and let the minimum edge weight be -10 (i.e., $\min_e\{w(e)\} = -10$). The following algorithm can be used to compute the shortest path between $u, v \in V$ in G: (i) Construct graph $G' = (V, E)$ with edge weights $w' : E \to \mathbb{R}_{\geq 0}$ defined as $w'(e) = w(e) + 10$, for all $e \in E$. (ii) Run Dijkstra's algorithm to compute the shortest path $P(u, v)$ between u, v in G'. (Note that by construction G' only has non-negative weight edges). (iii) Output $P(u, v)$ as the shortest path between u, v in G.

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐

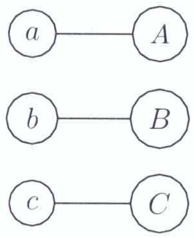
(d) Consider an undirected, connected, and simple graph $G = (V, E)$ with non-negative edge weights. Also consider a minimum spanning tree T of G. Then for every pair of vertices $u, w \in V$, the shortest (u, w)-path in G is contained in T.

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐

(e) Consider a connected [& simple] graph $G = (V, E)$ with 13 vertices (i.e., $|V| = 13$) such that G has an Euler path but no Euler tour. Then G must have exactly two vertices of odd degree and eleven vertices of even degree.

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐

(f) Consider the depicted matching M and corresponding preference lists. Then M is a stable matching.



$$B >_a A >_a C$$
$$C >_b B >_b A$$
$$A >_c C >_c B$$
$$b >_A a >_A c$$
$$c >_B b >_B a$$
$$a >_C c >_C b$$

[Reminder: $x >_z y$ indicates that z prefers to be matched with x over y.]

True . . . . ☐
False . . . . ☐
I prefer to not give an answer . . . . ☐