Exam Stochastic Models (theory)
Monday May 20, 13.45 - 16.45 hours.

Modelling and analysis of stochastic processes for Math (201400434, coord. Scheinhardt)
Modelling and analysis of stochastic processes for IEM (201400062, coord. Mes)

Teachers: Boucherie, Braaksma, Scheinhardt
Books, notes, etc. are **not** allowed. An ordinary (scientific) calculator is allowed, but a programmable or graphic calculator ('GR') is **not** allowed.

> This exam consists of **four exercises.**
> Use **different sheets of paper** for exercises 1–2 and exercises 3–4, respectively.
> Put your name and student number on **each paper** you submit.
> **Motivate** all answers.

Total points: 90. Grade = 1 + obtained points / 10

1. (24 points) Customers arrive at a service center according to a Poisson process with rate $\lambda = 4$ per hour. Customers have an exponential service time distribution with mean $1/\mu = 30$ minutes.

(a)–3 pt Argue that on the long run a single server is not sufficient for stability. How many servers would be required to have a stable system?

Instead of having more than one server available all the time, the service center decides to increase and decrease the number of available servers, depending on the number of customers present, as follows. There is one server if 0, 1, 2, or 3 customers are present, and there is one additional server from the back-office for each additional customer in the system up to a maximum of 4 servers (when 6 or more customers are present). Upon completion of service by a server, according to the same rule the service center decides whether or not this server will start serving a next customer, or returns to the back-office.

(b)–3 pt Let $X(t)$ be the number of customers in the system at time t. Why is $\{X(t),\ t > 0\}$ a Markov chain?

(c)–2 pt Give the distribution of an uninterrupted time during which the system contains exactly 10 customers. Motivate your answer.

(d)–3 pt Give the transition diagram of $\{X(t),\ t > 0\}$, including a motivation for the transition rates in this diagram.

(e)–4 pt Let $P(j)$, $j = 0, 1, 2, \ldots$, denote the steady state distribution of $X(t)$. Give the equations to determine this steady state distribution, and solve these equations to obtain the steady state distribution.

P.T.O.

The answers to the next questions may be expressed in λ, μ, and $P(j)$, $j = 0, 1, 2, \ldots$. [The solution of the system in question (e) is therefore not required for these answers.]

(f)–2 pt Give an expression for the average number of served customers leaving the system per unit time. Motivate your answer.

(g)–2 pt Give an expression for the average number of waiting customers (not including those that are being served). Motivate your answer.

(h)–2 pt Derive an expression for the mean waiting time (time in queue, before service starts). Motivate your answer.

(i)–3 pt Now assume that customers pay a fixed amount of €100 for service, but that waiting customers are offered food and drinks, which costs €20 per waiting customer per hour. The cost for a server that is not in the back-office is €50 per hour. Give an expression for the net profit of the service center.

2. (21 points) Consider a closed network of 4 stations containing m customers. Every station contains a single server, and all arriving customers arriving at a station may enter the station. Service is in order of arrival. The service times at stations 1, 2, 3, and 4 have exponential distributions with rates $\mu_1 = 4, \mu_2 = 3, \mu_3 = 2$, and $\mu_4 = 1$, respectively. The fraction p_{ij} of customers that are routed to station j upon departure from station i, $i, j = 1, 2, 3, 4$, is

$$p_{12} = 1,\ p_{23} = \frac{1}{3},\ p_{24} = \frac{2}{3},\ p_{34} = 1,\ p_{41} = \frac{1}{2},\ p_{42} = \frac{1}{2}.$$

(a)–4 pt Give the traffic equations (flow equations), and solve these equations.

(b)–5 pt Determine the average number of visits (to any of the stations) made by a customer after leaving station i, before arriving for the first time at station 3, $i = 1, 2, 3, 4$. Hint: describe the respective visits as a Markov chain in discrete time.

(c)–6 pt Give the joint stationary distribution for the number of customers in the four stations for $m = 1, 2$, including the normalisation constant.

(d)–6 pt Give the Mean Value Analysis algorithm, and use this algorithm to obtain the average number of customers and the average sojourn time in the four queues for $m = 1, 2$.

3. (20 points) Suppose that you have your own company, but in the next six months you want to focus all your time and energy on your study. Therefore, you need an employee to run your company for you during this period. You have found an employee, and at the beginning of each month you can decide on his salary for that month: either a low salary (€2300) or a high salary (€3000). Knowing his salary, the employee can decide to send in his resignation immediately. If the salary is low, the employee resigns with probability 0.4, while he resigns with probability 0.2 if the salary is high. If the employee quits, you have to hire a temporary employee immediately for €4000 per month. Once you have a temporary employee, the next month you start advertising for a new permanent employee. If you succeed in finding one, he will receive the same salary conditions as the original employee, and will start at the beginning of the following month (so each temporary employee stays for at least two months). The probability that you actually find a new permanent employee depends on the advertising budget. For a monthly advertising budget of €300, you will find a new permanent employee with probability 0.7; for an advertising budget of €600, this probability is 0.9. Hence, during each month that you start with a temporary employee, you have to choose the advertising budget. Your aim is to minimize the total costs over the next six months.

(a)–5 pt Formulate the problem as a stochastic dynamic program. What do you choose as stages, states, decisions and optimal value function?

(b)–6 pt What is the recurrence relation of the optimal value function?

(c)–6 pt Use dynamic programming to solve the problem. What are your expected minimum total costs?

(d)–3 pt What is the optimal policy? Describe it in a table, indicating for each stage and state the optimal decision.

P.T.O.