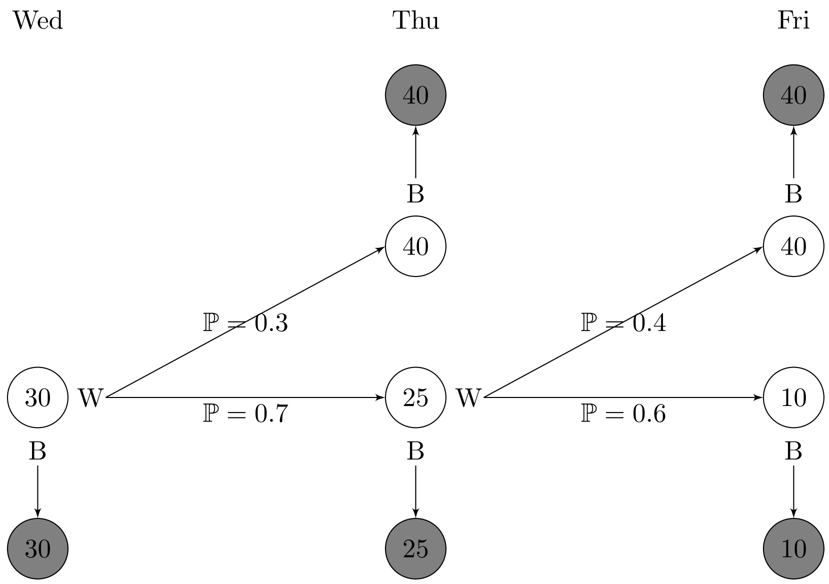
Answers Example Exam 1 Stochastic Models M8

1. **A store is known for is bargains. The store has the habit of lowering the price of its bargains each day, to ensure that articles are sold fast. Assume that you spot an item on Wednesday (there is only one of it left) that costs €30 and that you would like to buy for a friend as present for Saturday. You know that the price will be lowered to €25 on Thursday when the item is not sold, and to €l0 on Friday. You estimate that the probability that the item will be available on Thursday equals 0.7. You further estimate that the probability that it is still available on Friday when it was available on Thursday equals 0.6. You are sure that the item will no longer be available on Saturday. When you postpone your decision to buy the item to either Thursday or Friday, and the item is sold, you will buy another item of €40 as present for Saturday.**

   (a) **Formulate the problem as a stochastic dynamic programming problem. Specify phases, states, decisions and the optimal value function.**
   Phases: the days, $d = 1, 2, 3$ for Wednesday, Thursday, and Friday.
   States: the price of the cheapest present you can buy: $e \in \{10, 25, 30, 40\}$. (State $e = 40$ represents that the item at the bargain shop has been sold and you will have to buy the alternative gift.)
   Decisions: whether to buy the gift at the bargain shop (if it is still there) or wait, $a_d \in \{B, W\}$.
   Optimal value function: $V_d(e)$ = the minimal price you can expect to buy a gift for if it is day d and the current cheapest gift costs e.

   (b) **Draw the decision tree for this problem.**



   (c) **Give the recurrence relations for the optimal value function.**
   $V_3(e_3) = e_3$
   and for $d < 3$:
   $V_d(e_d) = \min_{a_d \in \{B,W\}} \left\{ c(e_d, a_d) + \sum_{e_{d+1}} \mathbb{P}(e_{d+1}|e_d, a_d) \cdot V_{d+1}(e_{d+1}) \right\}$
   With: $c(e, B) = e$ and $c(e, W) = 0$, and
   $\mathbb{P}(e_3 = 10|e_2 = 25, W) = 0.6$ $\quad \mathbb{P}(e_2 = 25|e_1 = 30, W) = 0.7$
   $\mathbb{P}(e_3 = 40|e_2 = 25, W) = 0.4$ $\quad \mathbb{P}(e_2 = 40|e_1 = 30, W) = 0.3$
   $\mathbb{P}(e_3 = 40|e_2 = 40, W) = 1$ $\quad \mathbb{P}(e_{d+1}|e_d, a_d) = 0$ otherwise

   (d) **What is the minimal expected amount that you will pay for your present, and what is the optimal decision on Wednesday?**
   Friday ($t = 3$):

$$
\begin{aligned}
V_3(40) &= 40 \rightarrow \text{buy} \\
V_3(10) &= 10 \rightarrow \text{buy}
\end{aligned}
$$

Thursday ($t = 2$):

$$\begin{aligned} V_1(40) &= 40 \to \text{buy} \\ V_1(25) &= \min\{25, 0 + 0.4 \cdot 40 + 0.6 \cdot 10\} \\ &= \min\{25, 22\} = 22 \to \text{wait} \end{aligned}$$

Wednesday ($t = 1$):

$$\begin{aligned} V_0(30) &= \min\{30, 0 + 0.3 \cdot 40 + 0.7 \cdot 22\} \\ &= \min\{30, 27.4\} = 27.4 \to \text{wait} \end{aligned}$$

So the minimal expected amount that you will pay for your present is €27.40, and the optimal decision on Wednesday is don't buy.

2. **Each day you own 0 or 1 stocks of a certain commodity. The price of the stock is a stochastic process that can be modeled as a Markov chain with transition rates as follows:**

	day $n+1$	
day n	100	200
100	0.50	0.50
200	0.25	0.75

At the start of a day at which you own a stock you may choose to either sell at the current price, or keep the stock. At the start of a day at which you do not own stock, you may choose to either buy one stock at the current price or do nothing. You have initial capital of 200. Your target is to maximize the discounted value of the profit over an infinite horizon, use discount factor 0.8 (per day).

(a) **Define the states and give for each state the possible decisions.**
States (i, j) with i the number of stock you own at the start of a day and j the current stock price, $(i, j) \in \{(0, 100), (0, 200), (1, 100), (1, 200)\}$.
Possible decisions for each state:
$D(0, 100) = D(0, 200) = \{\text{buy, do nothing}\} = \{B, N\}$,
$D(1, 100) = D(1, 200) = \{\text{sell, do nothing}\} = \{S, N\}$.

(b) **Formulate the optimality equations.**
$V(i, j) = \max_{d \in \{B,N,S\}} \left\{ r\Big((i, j), d\Big) + 0.8 \sum_{k,m} p\Big((k, m)|(i, j), d\Big) V(k, m) \right\}$
with
$r((0, 100), B) = -100$
$r((0, 200), B) = -200$
$r((1, 100), S) = 100$
$r((1, 200), S) = 200$
$r((i, j), N) = 0 \;\forall\; (i, j)$
and

$p\Big((k, m)|(i, j), N\Big)$:

	(k,m)			
(i, j)	$(0, 100)$	$(0, 200)$	$(1, 100)$	$(1, 200)$
$(0, 100)$	0.50	0.50	0	0
$(0, 200)$	0.25	0.75	0	0
$(1, 100)$	0	0	0.50	0.50
$(1, 200)$	0	0	0.25	0.75

$p\Big((k,m)|(i,j),S\Big)$:

(i,j)	(k,m) $(0,100)$	$(0,200)$
$(1,100)$	0.50	0.50
$(1,200)$	0.25	0.75

$p\Big((k,m)|(i,j),B\Big)$:

(i,j)	(k,m) $(1,100)$	$(1,200)$
$(0,100)$	0.50	0.50
$(0,200)$	0.25	0.75

(c) **Carry out the initialization and two additional iterations of value iteration.**

$$V_0(0,100) = \max\begin{cases} -100 & (d=B) \\ 0^* & (d=N) \end{cases}$$

$$V_0(0,200) = \max\begin{cases} -200 & (d=B) \\ 0^* & (d=N) \end{cases}$$

$$V_0(1,100) = \max\begin{cases} 100^* & (d=S) \\ 0 & (d=N) \end{cases}$$

$$V_0(1,200) = \max\begin{cases} 200^* & (d=S) \\ 0 & (d=N) \end{cases}$$

$$V_1(0,100) = \max\begin{cases} -100 + 0.8(0.5V_0(1,100) + 0.5V_0(1,200)) = 20^* & (d=B) \\ 0 + 0.8(0.5V_0(0,100) + 0.5V_0(0,200)) = 0 & (d=N) \end{cases}$$

$$V_1(0,200) = \max\begin{cases} -200 + 0.8(0.25V_0(1,100) + 0.75V_0(1,200)) = -60 & (d=B) \\ 0 + 0.8(0.25V_0(0,100) + 0.75V_0(0,200)) = 0^* & (d=N) \end{cases}$$

$$V_1(1,100) = \max\begin{cases} 100 + 0.8(0.5V_0(0,100) + 0.5V_0(0,200)) = 100 & (d=S) \\ 0 + 0.8(0.5V_0(1,100) + 0.5V_0(1,200)) = 120^* & (d=N) \end{cases}$$

$$V_1(1,200) = \max\begin{cases} 200 + 0.8(0.25V_0(0,100) + 0.75V_0(0,200)) = 200^* & (d=S) \\ 0 + 0.8(0.25V_0(1,100) + 0.75V_0(1,200)) = 140 & (d=N) \end{cases}$$

$$V_2(0,100) = \max\begin{cases} -100 + 0.8(0.5V_1(1,100) + 0.5V_1(1,200)) = 28^* & (d=B) \\ 0 + 0.8(0.5V_1(0,100) + 0.5V_1(0,200)) = 8 & (d=N) \end{cases}$$

$$V_2(0,200) = \max\begin{cases} -200 + 0.8(0.25V_1(1,100) + 0.75V_1(1,200)) = -56 & (d=B) \\ 0 + 0.8(0.25V_1(0,100) + 0.75V_1(0,200)) = 4^* & (d=N) \end{cases}$$

$$V_2(1,100) = \max\begin{cases} 100 + 0.8(0.5V_1(0,100) + 0.5V_1(0,200)) = 108 & (d=S) \\ 0 + 0.8(0.5V_1(1,100) + 0.5V_1(1,200)) = 128^* & (d=N) \end{cases}$$

$$V_2(1,200) = \max\begin{cases} 200 + 0.8(0.25V_1(0,100) + 0.75V_1(0,200)) = 204^* & (d=S) \\ 0 + 0.8(0.25V_1(1,100) + 0.75V_1(1,200)) = 144 & (d=N) \end{cases}$$

(d) **Formulate the LP model to solve this problem. Describe how you can obtain the optimal policy from the LP formulation.**

$$\begin{array}{llcll}
\min & z & = & V_{0,100}+V_{0,200}+V_{1,100}+V_{1,200} & \\
\text{s.t.} & V_{0,100} & \geq & -100+0.4V_{1,100}+0.4V_{1,200} & \text{State } (0,100) \quad \text{Decision } B \\
 & V_{0,100} & \geq & 0+0.4V_{0,100}+0.4V_{0,200} & \text{State } (0,100) \quad \text{Decision } N \\
 & V_{0,200} & \geq & -200+0.2V_{1,100}+0.6V_{1,200} & \text{State } (0,200) \quad \text{Decision } B \\
 & V_{0,200} & \geq & 0+0.2V_{0,100}+0.6V_{0,200} & \text{State } (0,200) \quad \text{Decision } N \\
 & V_{1,100} & \geq & 100+0.4V_{0,100}+0.4V_{0,200} & \text{State } (1,100) \quad \text{Decision } S \\
 & V_{1,100} & \geq & 0+0.4V_{1,100}+0.4V_{1,200} & \text{State } (1,100) \quad \text{Decision } N \\
 & V_{1,200} & \geq & 200+0.2V_{0,100}+0.6V_{0,200} & \text{State } (1,200) \quad \text{Decision } S \\
 & V_{1,200} & \geq & 0+0.2V_{1,100}+0.6V_{1,200} & \text{State } (1,200) \quad \text{Decision } N
\end{array}$$

The optimal policy can be obtained by taking the constraints that are binding and using the corresponding decisions in the corresponding states.

(e) **Choose a stationary policy and investigate using the policy iteration algorithm whether or not that policy is optimal.**

E.g., start with the stationary policy $\delta(0,100)=B,\ \delta(0,200)=\delta(1,100)=N,\ \delta(1,200)=S$. For this policy, the value determination equations are:

$$\begin{aligned}
V_\delta(0,100) &= -100+0.8\,[0.5V_\delta(1,100)+0.5V_\delta(1,200)] \\
V_\delta(0,200) &= 0+0.8\,[0.25V_\delta(0,100)+0.75V_\delta(0,200)] \\
V_\delta(1,100) &= 0+0.8\,[0.5V_\delta(1,100)+0.5V_\delta(1,200)] \\
V_\delta(1,200) &= 200+0.8\,[0.25V_\delta(0,100)+0.75V_\delta(0,200)]
\end{aligned}$$

Solving these equations we obtain $V_\delta(0,100)=50,\ V_\delta(0,200)=25,\ V_\delta(1,100)=150,\ V_\delta(1,200)=225$. The policy improvement step of the policy iteration methods yields

$$T_\delta(0,100)=\max\begin{cases}-100+0.8[0.5V_\delta(1,100)+0.5V_\delta(1,200)]=50^* & (d=B)\\ 0+0.8[0.5V_\delta(0,100)+0.5V_\delta(0,200)]=30 & (d=N)\end{cases}$$

$$T_\delta(0,200)=\max\begin{cases}-200+0.8[0.25V_\delta(1,100)+0.75V_\delta(1,200)]=-35 & (d=B)\\ 0+0.8[0.25V_\delta(0,100)+0.75V_\delta(0,200)]=25^* & (d=N)\end{cases}$$

$$T_\delta(1,100)=\max\begin{cases}100+0.8[0.5V_\delta(0,100)+0.5V_\delta(0,200)]=130 & (d=S)\\ 0+0.8[0.5V_\delta(1,100)+0.5V_\delta(1,200)]=150^* & (d=N)\end{cases}$$

$$T_\delta(1,200)=\max\begin{cases}200+0.8[0.25V_\delta(0,100)+0.75V_\delta(0,200)]=225^* & (d=S)\\ 0+0.8[0.25V_\delta(1,100)+0.75V_\delta(1,200)]=165 & (d=N)\end{cases}$$

For each state (i,j), we have $T_\delta(i,j)=V_\delta(i,j)$. Thus, δ is an optimal stationary policy.

(f) **Give the number of stationary policies. Motivate your answer by using the definition of stationary policy.**

A policy δ is a stationary policy if whenever the state is (i,j), the policy δ chooses (independently of the period) the same decision. In this case, there are four different states and there are two possible decisions in each state. Hence, the number of stationary policies is $2^4=16$.

3. **Consider a queueing system with 1 counter, to which groups of customers arrive according to a Poisson process with intensity λ. The size of a group is 1 with probability p and 2 with probability $1-p$. Customers are served one by one. The service time has exponential distribution with mean μ^{-1}. Service times are mutually independent and independent of the arrival process. The system may contain at most 3 customers. If the system is full upon arrival of a group, or if the system may contain only one additional customer upon arrival of a group of size 2, then all customers in the group are lost and will never return. Let $Z(t)$ record the number of customers at time t.**

(a) **Explain why $\{Z(t), t \geq 0\}$ is a Markov process and give the diagram of transitions and transition rates.**
$\{Z(t), t \geq 0\}$ is a Markov process because the future number of customers only depends on the number of customers in the system at time t, not on the evolution of the number of customers in the system before time t. This is due to fact that the interarrival times as well as the service times are exponentially distributed, so both distributions are memoryless. Therefore, if the last event (an arrival or a departure) happened at time $t - h$ and it is now time t, the time from now until the next event depends only on $Z(t)$, the current state of the system. Figure **??** displays the transition diagram.

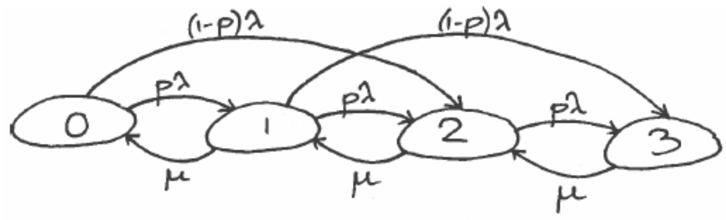


Figure 1: Transition diagram 6a.

(b) **Give the equilibrium equations (balance equations) for the stationary probabilities $P_n, n = 0, 1, 2, 3$.**

$$\begin{aligned}
(1-p)\lambda P_0 + p\lambda P_0 &= \mu P_1 && (1)\\
(1-p)\lambda P_1 + p\lambda P_1 + \mu P_1 &= p\lambda P_0 + \mu P_2 && (2)\\
p\lambda P_2 + \mu P_2 &= (1-p)\lambda P_0 + p\lambda P_1 + \mu P_3 && (3)\\
\mu P_3 &= (1-p)\lambda P_1 + p\lambda P_2 && (4)\\
P_0 + P_1 + P_2 + P_3 &= 1 && (5)
\end{aligned}$$

(c) **Compute these probabilities $P_n, n = 0, 1, 2, 3$.**

$$\begin{aligned}
(\mathbf{??}) : \lambda P_0 &= \mu P_1\\
P_1 &= \frac{\lambda}{\mu} P_0 = \rho P_0\\
(\mathbf{??}) : (\lambda + \mu) P_1 &= p\lambda P_0 + \mu P_2\\
\mu P_2 &= (\lambda + \mu)\rho P_0 - p\lambda P_0\\
P_2 &= \frac{\lambda + \mu}{\mu}\rho P_0 - p\rho P_0\\
P_2 &= (\rho + 1 - p)\rho P_0\\
(\mathbf{??}) : \mu P_3 &= (1-p)\lambda\rho P_0 + p\lambda(\rho + 1 - p)\rho P_0\\
\mu P_3 &= (1 + p\rho - p^2)\lambda\rho P_0\\
P_3 &= (1 + p\rho - p^2)\rho^2 P_0\\
(\mathbf{??}) : 1 &= P_0 + \rho P_0 + (\rho + 1 - p)\rho P_0 + (1 + p\rho - p^2)\rho^2 P_0\\
1 &= \Big(1 + (2-p)\rho + (2-p^2)\rho^2 + p\rho^3\Big) P_0\\
P_0 &= \Big(1 + (2-p)\rho + (2-p^2)\rho^2 + p\rho^3\Big)^{-1}
\end{aligned}$$

The answers to the following questions may be provided in terms of the probabilities P_n (except for (h)).

(d) **Give an expression for the average number of waiting customers.**
$\mathbb{E}(L_q) = P_2 + 2P_3$

(e) **Give the departure rate and the rate at which customers *enter* the system.**
Customers enter the system at rate $1 \cdot p\lambda(P_0 + P_1 + P_2) + 2 \cdot (1-p)\lambda(P_0 + P_1)$.
This equals the departure rate: $\mu(P_1 + P_2 + P_3) = \mu(1 - P_0)$.

(f) **Give an expression for the average waiting time of a customer.**
Little: $\mathbb{E}(L_q) = \lambda\mathbb{E}(W_q) \Rightarrow \mathbb{E}(W_q) = \frac{1}{\lambda}\mathbb{E}(L_q)$.

In Little's law, λ is the rate at which customers actually *enter* the system, which is $\lambda(P_0 + P_1) + p\lambda P_2$ in this case. Hence

$$\mathbb{E}(W_q) = \frac{P_2 + 2P_3}{\lambda(P_0 + P_1) + p\lambda P_2} = \frac{P_2 + 2P_3}{\mu(1 - P_0)}.$$

(g) **What is the fraction of time the counter is busy?**
$1 - P_0$

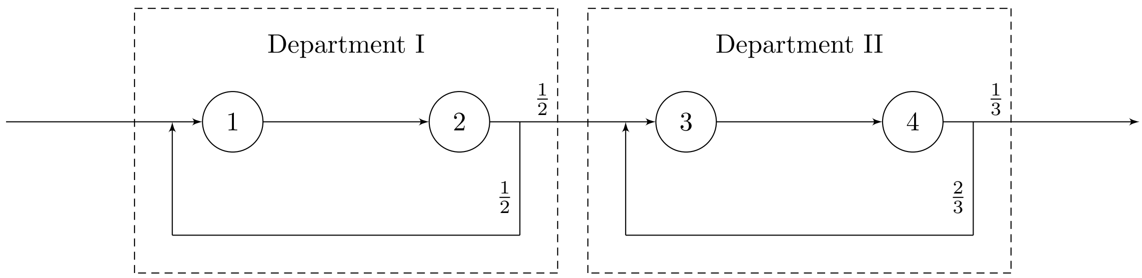
(h) **What is the average length of an idle period?**
$\frac{1}{\lambda}$

(i) **Determine from (g) and (h) the average length of a period the system is occupied (=at least 1 customer in the system).**
(A busy period is an uninterrupted interval in which the server is busy. A busy cycle is the time between two consecutive moments at which a busy period starts.)
During a busy cycle, the counter is on average busy a fraction $1 - P_0$ of the time and idle a fraction P_0 of the time (g). The average length of the idle period is $\frac{1}{\lambda}$ (h). Hence, the average length of the busy period is

$$\frac{1 - P_0}{P_0} \cdot \frac{1}{\lambda} = \frac{1 - P_0}{\lambda P_0}.$$

(j) **What is the rate at which groups of size 2 *enter* the system?**
$(1 - p)\lambda(P_0 + P_1)$

4. **Consider the open network in the figure underneath.**



The queueing system consists of four queues: 1, 2, 3, and 4. Queues 1 and 2 constitute Department I, Queues 3 and 4 constitute Department II. The numbers alongside the arrows represent the transition probabilities between the four stations. For example, a customer departing from Station 4 goes to Station 3 with probability 2/3, and leaves the network with probability 1/3. Each station has one server, and every arriving customer can be added to the queue. Service happens in order of arrival. Service times have exponential distributions with means $1/\mu_1 = 1/4$, $1/\mu_2 = 1/3$, $1/\mu_3 = 1/2$, $1/\mu_4 = 1$. The arrival rate at Station 1 is γ_1 (Poisson). [Note: queue i refers to the system consisting of the waiting room plus the server, $i = 1, 2, 3, 4$.]

(a) **Formulate the traffic equations and solve these equations.**

$$\begin{aligned}
\lambda_1 &= \gamma_1 + 0.5\lambda_2 \\
\lambda_2 &= \lambda_1 \\
\lambda_3 &= 0.5\lambda_2 + \frac{2}{3}\lambda_4 \\
\lambda_4 &= \lambda_3
\end{aligned}$$

Solution: $\lambda_2 = \lambda_1 = 2\gamma_1$, $\lambda_4 = \lambda_3 = 3\gamma_1$.

(b) **What is the stability condition?**
$\max_i\{\rho_i\} = \max_i\{\lambda_i/\mu_i\} < 1$, so $\gamma_1 < 1/3$.

(c) **Give the probability distribution of the number of customers at Stations 1, 2, 3, and 4.**

The probability that there are n_i customers at Station i equals $\pi_i(n_i) = (1-\rho_i)(\rho_i)^{n_i}$:

$$\begin{aligned}
\pi_1(n_1) &= \left(1 - \frac{2\gamma_1}{4}\right)\left(\frac{2\gamma_1}{4}\right)^{n_1} \\
\pi_2(n_2) &= \left(1 - \frac{2\gamma_1}{3}\right)\left(\frac{2\gamma_1}{3}\right)^{n_2} \\
\pi_3(n_3) &= \left(1 - \frac{3\gamma_1}{2}\right)\left(\frac{3\gamma_1}{2}\right)^{n_3} \\
\pi_4(n_4) &= (1 - 3\gamma_1)(3\gamma_1)^{n_4}
\end{aligned}$$

(d) **Give the joint distribution of the queue lengths at the stations (product form).**
$\pi(n_1, n_2, n_3, n_4) = \pi_1(n_1) \cdot \pi_2(n_2) \cdot \pi_3(n_3) \cdot \pi_4(n_4)$

(e) **For each station, give the average number of customers in the queue, and the average sojourn time of a customer at that queue.**
The average number of customers in the queue at Station i is $\mathbb{E}(L_{q,i}) = \rho_i^2/(1-\rho_i) = \lambda_i^2/(\mu_i(\mu_i - \lambda_i))$:

$$\mathbb{E}(L_{q,1}) = \frac{\gamma_1^2}{4 - 2\gamma_1},\ \mathbb{E}(L_{q,2}) = \frac{4\gamma_1^2}{9 - 6\gamma_1},\ \mathbb{E}(L_{q,3}) = \frac{9\gamma_1^2}{4 - 6\gamma_1},\ \mathbb{E}(L_{q,4}) = \frac{3\gamma_1^2}{1 - 3\gamma_1}.$$

By Little's law, the mean waiting time at Station i is $\mathbb{E}(W_{q,i}) = \mathbb{E}(L_{q,i})/\lambda_i$, so the mean sojourn time at Station i equals $\mathbb{E}(W_i) = \mathbb{E}(W_{q,i}) + 1/\mu_i = 1/(\mu_i - \lambda_i)$:

$$\mathbb{E}(W_1) = \frac{1}{4 - 2\gamma_1},\ \mathbb{E}(W_2) = \frac{1}{3 - 2\gamma_1},\ \mathbb{E}(W_3) = \frac{1}{2 - 3\gamma_1},\ \mathbb{E}(W_4) = \frac{1}{1 - 3\gamma_1}.$$

(f) **Give an expression for the average sojourn time in Department II.**
$(\mathbb{E}(L_3) + \mathbb{E}(L_4))/\gamma_1$, because γ_1 is the number of customers truly entering the system (and thus also Department II) per time unit.

$$\frac{\mathbb{E}(L_3) + \mathbb{E}(L_4)}{\gamma_1} = \frac{\frac{3\gamma_1}{2-3\gamma_1} + \frac{3\gamma_1}{1-3\gamma_1}}{\gamma_1} = \frac{3}{2 - 3\gamma_1} + \frac{9}{1 - 3\gamma_1}.$$

Alternative route to the same answer:
The mean sojourn time for *a single visit* to Department II is:

$$\mathbb{E}(W_3) + \mathbb{E}(W_4) = \frac{1}{2 - 3\gamma_1} + \frac{1}{1 - 3\gamma_1}$$

Note that the number of times a customer visits Department II before leaving the system is geometrically distributed with 'success probability' $p = \frac{1}{3}$. Hence, the average number of times a customer visits Department II equals $\frac{1}{1/3} = 3$. Concluding, a customer's *total expected sojourn time* in Department II equals:

$$3 \cdot \left(\frac{1}{2 - 3\gamma_1} + \frac{1}{1 - 3\gamma_1}\right) = \frac{3}{2 - 3\gamma_1} + \frac{3}{1 - 3\gamma_1}.$$