Exam Markov chains, part 2, module 8 (201400434)
Wednesday 17 June 2015, 8.45 - 11.45
Coordinator and teacher: W.R.W. Scheinhardt

This exam consists of 3 exercises.
Using a simple (non-graphic) calculator is allowed
Using the book is not allowed
Motivate your answers

1. We assume that earthquakes on Java occur according to a Poisson process with intensity 0.8 per year. We distinguish between 'minor' and 'major' earthquakes. The probability that an earthquake is 'minor' is 0.95 and independent of other circumstances.

   a. Give one of the definitions of a Poisson process with intensity λ.

   b. Determine the probability that in the period 2016 up to and including 2018 exactly one major quake and three minor quakes will occur.

   c. We now live in June 2015. In what year do you expect the first major earthquake, given that the previous one occurred in April 2011?

   d. Do you think the assumption that earthquakes occur according to a Poisson process is justified? Why or why not?

2. Consider a discrete-time Markov chain $\{X_n\}$.

   a. The notation $i \leftrightarrow j$ means that i and j communicate. Prove that if $i \leftrightarrow j$ and $j \leftrightarrow k$, then also $i \leftrightarrow k$.

   b. Can a *steady-state* distribution exist if $\{X_n\}$ is irreducible, positive recurrent, and *periodic*? Motivate your answer.

   Now let X_n be the capital of a gambler after the n^{th} game and let X_0 be his initial capital. As long as the gambler has money left, he bets 1 euro per game, after which he obtains 0 or 2 euros with probabilities 0.6 and 0.4 respectively. When the gambler has no money left, he stops playing, i.e. $X_n = 0 \Rightarrow X_{n+k} = 0$ for $k = 1, 2, \dots$. Assume that $\{X_n\}$ is a Markov chain, and that $X_0 = 10$.

   c. Determine the communicating classes of the chain $\{X_n\}$, determine the period of each class and verify for each class whether it is transient, null recurrent, or positive recurrent.

   d. Determine the limiting probabilities $\lim_{n\to\infty} P(X_n = i|X_0 = 10)$ for $i = 0, 1, \dots$.

   e. Give the system of equations with which the probability can be determined that the gambler will ever own 11 euros, before he goes bankrupt (you need not solve it).

3. In a warehouse, a certain type of product is held in stock. At times which are generated by a Poisson process with intensity 2, exactly one item is removed from the warehouse, as long as the stock lasts. As soon as the stock has come down to one item only, three new items are ordered which are delivered only after an exponentially distributed time with expectation 1. Let $X(t)$ be the stock size at time t and assume that $X(0) = 3$.
   a. Determine the generator matrix Q of the Markov chain $\{X(t)\}$.
   b. The system of Kolmogorov backward equations consists of 16 linked differential equations for the transition probability functions $P_{ij}(t)$. Give (only) the equation for $P'_{1,2}(t) = \ldots$.
   c. Apply uniformisation to the Markov chain $\{X(t)\}$ using a suitable uniformisation rate, and give the transition matrix P^* of the uniformised DTMC (discrete time Markov chain).

Grading:

1				2					3			Total
a	b	c	d	a	b	c	d	e	a	b	c	
2	3	2	1	3	2	3	2	2	2	2	3	27

$$\text{Grade} = \frac{\text{points achieved}}{3} + 1 \quad \text{(rounded to 1 decimal)}$$