Exam Stochastic Models (theory)
Wednesday June 10, 2020, 8.45 - 11.45 hours.

Modelling and analysis of stochastic processes for Math (201400434, coord. Scheinhardt)
Modelling and analysis of stochastic processes for IEM (201400062, coord. Mes)

Teachers: Boucherie, Timmer, Scheinhardt

This exam consists of **four exercises**.
Please start each exercise on a new sheet of paper.
Put your name and student number on **each sheet** you submit.
Motivate all answers
Use your time **efficiently**; e.g., correct miscalculations in time-consuming solutions only when you have time left; most points are awarded for showing your understanding.

Total points: 90. Grade = 1 + obtained points / 10

Integrity statement

Please read the following paragraph carefully, copy the text below it verbatim to the first page of your work (handwritten), add your full name and program (IEM/Math/...) and sign it. If you fail to do so, your test will not be graded.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

The **only allowed sources** for this test are:

- the books by Winston and/or Ross as used in this module (hardcopy or pdf)
- the slides (printed or pdf)
- your own summaries/notes (but no solutions to tutorial/exam problems).
- an ordinary (scientific) calculator (not a programmable or graphic calculator)
- other electronic devices (laptop/tablet/mobile phone), but only to be used:
  - for downloading the test
  - to show the test/book/slides on your screen
  - to write the test (in case you prefer to use a tablet instead of paper to write on)
  - for making scans or photos of your work and uploading them to Canvas

Copy this text (handwritten), add your full name and program, and sign it:

I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.

1. **(25 points)** Customers arrive at a service center according to a Poisson process with rate λ per hour. The service center has 2 queues, queue 1 and queue 2. The service center has 1 server available that may either serve a customer from queue 1 or from queue 2. All service times are exponentially distributed with service rate μ per hour. If the queues are not of equal length, then an arriving customer will always choose to join the shortest queue; if both queues are of equal length, then an arriving customer will always join queue 1. If the queues are not of equal length, then the server will always serve a customer from the longest queue; if the queues are of equal length then the server will serve a customer from queue 1; to achieve this policy, if needed the server will switch from serving a customer at queue 2 to serving a customer at queue 1. When the server later returns to the abandoned customer at queue 2, this customer will require a new ('fresh') full service time. Let $X(t)$ be the stochastic process describing the number of customers in both queues at time t; we denote its states by (n_1, n_2), where $n_i \in \{0, 1, 2, \ldots\}$ is the number in queue i, $i = 1, 2$.

   (a)–3 pt Why is $\{X(t),\ t > 0\}$ a Markov chain?

   (b)–3 pt Assume that $X(t)$ is in state (1,1). What is the distribution of the time until $X(t)$ leaves state $(1, 1)$? What is the probability that a service in queue 2 starts before a customer arrives to the system? Motivate your answer.

   (c)–2pt Which states are in the same communicating class as state $(0, 0)$? Motivate your answer.

   (d)–2 pt Give the stability condition for $\{X(t),\ t > 0\}$. Motivate your answer.

   (e)–4 pt Give the transition diagram of $\{X(t),\ t > 0\}$, including a motivation for the transition rates in this diagram.

   (f)–3 pt Let $P(n_1, n_2)$ denote the steady state distribution of $X(t)$. Give the equations to determine this steady state distribution.

   (g)–4 pt Show that $P(1, 1) = \left(\frac{\lambda}{\mu}\right)^2 P(0, 0)$.

   The following may be answered in terms of the $P(n_1, n_2)$ and the parameters λ and μ.

   (h)–2 pt What is the utilisation of the server? Motivate your answer.

   (i)–2 pt Give an expression for the average number of customers in queue 2. Motivate your answer.

P.T.O.

2. **(20 points)** Consider a closed network of 3 stations containing $m = 3$ customers. Every station contains a single server, and all arriving customers arriving at a station may enter the station. Service is in order of arrival. The service times at stations 1, 2, and 3 have exponential distributions with rates $\mu_1 = 3/8, \mu_2 = 3/16$, and $\mu_3 = 1/12$, respectively. The fraction r_{ij} of customers that is routed to station j upon departure from station i, $i, j = 1, 2, 3$, is determined by

$$r_{12} = 1,\ r_{21} = \frac{1}{3},\ r_{23} = \frac{2}{3},\ r_{31} = 1.$$

(a)–4 pt Give the traffic equations, and solve these equations.

(b)–2 pt Why is a stability condition not required for this network?

(c)–5 pt Give the equilibrium distribution, including the normalising constant. (Recall that $m = 3$.)

(d)–3 pt What is the probability that a customer arriving to station 3 is served immediately? Motivate your answer.

(e)–6 pt Determine the mean waiting time of the customers in station 3. Motivate your answer.

3. **(23 points)** For emergency reasons four doctors are to be sent to three different hospitals to help taking care of the patients there. The table below shows the number of patients the doctors can take care of per hour for each of the hospitals.

Number of doctors	Hospital 1	Hospital 2	Hospital 3
0	0	0	0
1	4	2	5
2	6	4	7
3	9	7	8
4	10	11	9

How should we allocate the doctors so that the number of patients that is taken care of per hour is as large as possible?

(a)–6 pt This problem can be formulated as a stochastic dynamic program. What do you choose as stages, states, decisions and optimal value function?

(b)–4 pt What is the recurrence relation of the optimal value function (including boundary conditions)?

(c)–9 pt Use dynamic programming to solve the problem. What is the largest number of patients that is taken care of per hour? How many doctors should be sent to each of the hospitals?

Suppose now the doctors should also help a fourth hospital. The number of patients the doctors can take care of per hour is shown in the following table.

Number of doctors	Hospital 4
0	0
1	3
2	5
3	8
4	10

(d)–2 pt What should change to your formulation of the stochastic dynamic program in (a)?

(e)–2 pt Describe how to solve this problem effectively, that is, using your solution to (c). (You need not solve it.)

4. **(22 points)** The start-up company LOT trades oaken tables that are handmade by a local carpenter. The company uses a part of a showroom that is just large enough to show two oaken tables. There is no warehouse available, so the company may have at most two oaken tables. Currently, they have no tables in the showroom.

   The demand per week is uncertain. Based on a market survey, the company expects to sell no tables with probability 0.3, one table with probability 0.5 and two tables with probability 0.2. If the demand of a week exceeds the number of tables available, the shortage is delivered immediately. This costs LOT 450 euros per table. Inventory costs are 10 euros per table per week.

   At the end of the week LOT has to decide how many tables to order, if any. Each order has a fixed cost of 50 euros plus 200 euros per ordered table. The order is delivered immediately.

   LOT wishes to minimize its expected discounted cost over an infinite horizon at discount factor 0.9 per week.

   (a)–4 pt Model this problem as a Markov decision problem. What do you choose as states, decisions and optimal value function?

   (b)–4 pt Determine the expected direct costs for each state and decision, as well as the transition probabilities.

   (c)–4 pt Formulate the linear programming model that LOT may use to determine the optimal cost.

   (d)–3 pt Suppose LOT's strategy is to order the largest number of tables possible in each state. Which equations does LOT have to solve to determine the expected discounted costs of this strategy?

   (e)–4 pt Perform two iterations of the value iteration algorithm. (So, starting from $V_0(i) = 0$, determine $V_1(i)$ and $V_2(i)$.) In each iteration indicate the corresponding best strategy.

   (f)–3 pt Now suppose LOT wants to quit the business, and stops ordering new tables. When two tables are in the showroom, what is the expected time till both are sold?